Topics of Algebra used in Calculus

<u>Factoring</u>

Factoring Polynomials:

$24x^2y^3 - 48xy^4 = (24xy^3)x - (24xy^3)2y = 24xy^3(x - 2y)$

$$\begin{aligned} x^2 - 12x + 36 &= x^2 - 6x - 6x + 36 = (x^2 - 6x) + (-6x + 36) \\ &= x(x-6) - 6(x-6) = (x-6)(x-6) = (x-6)^2 \end{aligned}$$

$$\begin{aligned} 2x^2 - 5x - 3 &= 2x^2 - 6x + x - 3 = (2x^2 - 6x) + (x - 3) \\ &= 2x(x-3) + 1(x-3) = (x-3)(2x+1) \end{aligned}$$

Quadratic Formula for $ax^2 + bx + c = 0$

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

If x_1 and x_2 are solutions ; then the factors are $(x - x_1)(x - x_2)$

Algebra Identities:

$(a+b)^2 = a^2 + 2ab + b^2$
$(a-b)^2 = a^2 - 2ab + b^2$
$(a+b)(a-b) = a^2 - b^2$

Example: To factor the expression

$9x^4 + 24x^2 + 16 = (3x^2)^2 + 2(3x^2)(4) + (4)^2 = (3x^2 + 4)^2$

Solve Algebra Equation:

Example 1: Solve for x : $4x(10-2x)-(10-2x)^2=0$

Factoring:
$$(10-2x)\,[4x-(10-2x)]=0$$
$$(10-2x)\,[4x-10+2x]=0$$
$$(10-2x)\,(6x-10)=0$$

$$10-2x=0 \implies x=\frac{10}{2}=5$$

$$6x-10=0 \implies x=\frac{10}{6}=\frac{5}{3}$$

Solution is $\left\{\frac{5}{3}\,,\,5\right\}$

Note: Always reduce your answer.
Never give decimal answer unless in application problem.

Example 2: Solve for x : $3x^2-6x+1=0$

Factoring is not possible.

It is a quadratic equation, must use the quadratic formula:

$$ax^2+bx+c=0 \implies x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-(-6)\pm\sqrt{(-6)^2-4(3)(1)}}{2(3)}=\frac{6\pm\sqrt{24}}{6}$$

$$x=\frac{6\pm2\sqrt{6}}{6}=\frac{2\left(3\pm\sqrt{6}\right)}{6}=\frac{3\pm\sqrt{6}}{3}$$

Solution is $x=\left\{\frac{3-\sqrt{6}}{3}\,,\,\frac{3+\sqrt{6}}{3}\right\}$

If required, the answer in decimals $x=\{0.18\,,\,1.82\}$

Solve Algebra Equation:

Example 3: Solve for a : $4x^3 + x^2(3ya) + 2xy^2 - 6ya = 0$

Group the terms with a on left side of equation and the other terms on right hand side of equation.
(don't forget in addition or subtraction, change sign when change side of equation)

$3x^2ya - 6ya = -4x^3 - 2xy^2$

Factor each side of equation:

$$3ya(x^2-2) = -2x(2x^2+y^2) \implies a = \frac{-2x(2x^2+y^2)}{3y(x^2-2)}$$

Note: Always reduce your answer.
Never give decimal answer unless in application problem.

Fractions or Rational Expressions: Common denominator

$$\frac{2}{x} + \frac{3}{x+1} = \frac{2(x+1)}{x(x+1)} + \frac{3x}{x(x+1)} = \frac{2(x+1)+3x}{x(x+1)} = \frac{2x+2+3x}{x(x+1)} = \frac{5x+2}{x(x+1)}$$

$$\frac{x}{x-1} - \frac{2}{x+3} = \frac{x(x+3)}{(x-1)(x+3)} - \frac{2(x-1)}{(x-1)(x+3)} = \frac{x(x+3)-2(x-1)}{(x-1)(x+3)} = \frac{x^2+3x-2x+2}{(x-1)(x+3)} = \frac{x^2+x+2}{(x-1)(x+3)}$$

$$\frac{5x}{2/x} + \frac{6}{x} = \frac{5x^2}{2} + \frac{6}{x} = \frac{5x^2(x)}{2x} + \frac{6(2)}{2x} = \frac{5x^3+12}{2x}$$

$$\frac{4}{x-2} - \frac{3}{(x-2)^2} = \frac{4(x-2)}{(x-2)^2} - \frac{3}{(x-2)^2} = \frac{4(x-2)-3}{(x-2)^2} = \frac{4x-8-3}{(x-2)^2} = \frac{4x-11}{(x-2)^2}$$

$$\frac{x^2-x+1}{(x+1)^2} + \frac{5}{x+1} = \frac{x^2-x+1}{(x+1)^2} + \frac{5(x+1)}{(x+1)^2} = \frac{x^2-x+1+5(x+1)}{(x+1)^2} = \frac{x^2-x+1+5x+5}{(x+1)^2} = \frac{x^2+4x+6}{(x+1)^2}$$

$$\frac{3}{x} - \frac{2}{(x+1)^2} = \frac{3(x+1)^2}{x(x+1)^2} - \frac{2x}{x(x+1)^2} = \frac{3(x^2+2x+1)-2x}{x(x+1)^2} = \frac{3x^2+6x+3-2x}{x(x+1)^2} = \frac{3x^2+4x+3}{x(x+1)^2}$$

$$\frac{\frac{1}{x} - \frac{2}{x+1}}{x} = \frac{\frac{1(x+1)}{x(x+1)} - \frac{2x}{x(x+1)}}{x} = \frac{1(x+1)-2x}{x(x+1)} \cdot \frac{1}{x} = \frac{x+1-2x}{x(x+1)} \cdot \frac{1}{x} = \frac{-x+1}{x^2(x+1)}$$

$$\frac{\frac{2}{x-1} + \frac{4}{x-2}}{x+3} = \frac{\frac{2(x-2)}{(x-1)(x-2)} + \frac{4(x-1)}{(x-1)(x-2)}}{x+3} = \frac{2(x-2)+4(x-1)}{(x-1)(x-2)} \cdot \frac{1}{x+3}$$

$$= \frac{2x-4+4x-4}{(x-1)(x-2)} \cdot \frac{1}{x+3} = \frac{6x}{(x-1)(x-2)(x-3)}$$

Fractions or Rational Expressions: Rationalize Numerator

Rationalize numerator when radical expressions are in the numerator:

$$\frac{\sqrt{x+1}-2}{x}=\frac{\left(\sqrt{x+1}-2\right)\left(\sqrt{x+1}+2\right)}{x\left(\sqrt{x+1}+2\right)}=\frac{x+1-4}{x\left(\sqrt{x+1}+2\right)}=\frac{x-3}{x\left(\sqrt{x+1}+2\right)}$$

$$\frac{\sqrt{x+9}-3}{x}=\frac{\left(\sqrt{x+9}-3\right)\left(\sqrt{x+9}+3\right)}{x\left(\sqrt{x+9}+3\right)}=\frac{x+9-9}{x\left(\sqrt{x+9}+3\right)}=\frac{x}{x\left(\sqrt{x+9}+3\right)}=\frac{1}{\sqrt{x+9}+3}$$

$$\frac{\sqrt{3x+4}-2}{x}=\frac{\left(\sqrt{3x+4}-2\right)\left(\sqrt{3x+4}+2\right)}{x\left(\sqrt{3x+4}+2\right)}=\frac{3x+4-4}{x\left(\sqrt{3x+4}+2\right)}=\frac{3x}{x\left(\sqrt{3x+4}+2\right)}=\frac{3}{\sqrt{3x+4}+2}$$

$$\frac{\sqrt{x+7}-3}{x-2}=\frac{\left(\sqrt{x+7}-3\right)\left(\sqrt{x+7}+3\right)}{(x-2)\left(\sqrt{x+7}+3\right)}=\frac{x+7-9}{(x-2)\left(\sqrt{x+7}+3\right)}=\frac{x-2}{(x-2)\left(\sqrt{x+7}+3\right)}=\frac{1}{\sqrt{x+7}+3}$$

$$\frac{\sqrt{8-x}-3}{x+1}=\frac{\left(\sqrt{8-x}-3\right)\left(\sqrt{8-x}+3\right)}{(x+1)\left(\sqrt{8-x}+3\right)}=\frac{8-x-9}{(x+1)\left(\sqrt{8-x}+3\right)}=\frac{-x-1}{(x+1)\left(\sqrt{8-x}+3\right)}$$

$$=\frac{-(x+1)}{(x+1)\left(\sqrt{8-x}+3\right)}=\frac{-1}{\sqrt{8-x}+3}$$

Fractions or Rational Expressions: Partial Fractions with Linear Denominator

Given a rational expression, split it into simple fractional expressions (linear denominator like $ax+b$):

Given $\dfrac{-x-2}{x(x-1)}$, find two partial fractions $\dfrac{A}{x}+\dfrac{B}{x-1}$

therefore $\dfrac{A}{x}+\dfrac{B}{x-1}=\dfrac{A(x-1)}{x(x-1)}+\dfrac{Bx}{x(x-1)}=\dfrac{A(x-1)+Bx}{x(x-1)}$ so

$A(x-1)+Bx=-x-2$; to solve for A and B:

Let $x=1 \implies A(1-1)+B(1)=-1-2 \implies B=-3$

Let $x=0 \implies A(0-1)+B(0)=-0-2 \implies A=2$

therefore $\dfrac{-x-2}{x(x-1)}=\dfrac{2}{x}+\dfrac{-3}{x-1}$

Fractions or Rational Expressions: Partial Fractions with Linear Denominator

Given a rational expression, split it into simple fractional expressions (linear denominator like $ax+b$):

Given $\dfrac{7x+11}{(x+1)(x+2)}$, find two partial fractions $\dfrac{A}{x+1}+\dfrac{B}{x+2}$

therefore $\dfrac{A}{x+1}+\dfrac{B}{x+2}=\dfrac{A(x+2)}{(x+1)(x+2)}+\dfrac{B(x+1)}{(x+1)(x+2)}=\dfrac{A(x+2)+B(x+1)}{(x+1)(x+2)}$ so

$A(x+2)+B(x+1)=7x+11$; to solve for A and B:

Let $x=-2 \implies A(-2+2)+B(-2+1)=7(-2)+11 \implies B=3$

Let $x=-1 \implies A(-1+2)+B(-1+1)=7(-1)+11 \implies A=4$

therefore $\dfrac{7x+11}{(x+1)(x+2)}=\dfrac{4}{x+1}+\dfrac{3}{x+2}$

Fractions or Rational Expressions: Partial Fractions with repeated Linear Denominator

Given a rational expression with repeated linear denominators like $(ax+b)^n$,
split it into simple fractional expressions :

Given $\dfrac{2x-7}{(x-1)^2}$, find two partial fractions $\dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}$

therefore $\dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}=\dfrac{A(x-1)}{(x-1)^2}+\dfrac{B}{(x-1)^2}=\dfrac{A(x-1)+B}{(x-1)^2}$ so

$A(x-1)+B=2x-7$; to solve for A and B:

Let $x=1 \implies A(1-1)+B=2(1)-7 \implies B=-5$

Let $x=0$, $B=-5 \implies A(0-1)+(-5)=2(0)-7 \implies A=2$

therefore $\dfrac{2x-7}{(x-1)^2}=\dfrac{2}{x-1}+\dfrac{-5}{(x-1)^2}$

Fractions or Rational Expressions: Partial Fractions with repeated Linear Denominator

Given a rational expression with repeated linear denominators like $(ax+b)^n$,
split it into simple fractional expressions :

Given $\dfrac{5x^2+13x+3}{x(x+1)^2}$, find three partial fractions $\dfrac{A}{x}+\dfrac{B}{x+1}+\dfrac{C}{(x+1)^2}$

therefore $\dfrac{A}{x}+\dfrac{B}{x+1}+\dfrac{C}{(x+1)^2}=\dfrac{A(x+1)^2}{x(x+1)^2}+\dfrac{Bx(x+1)}{x(x+1)^2}+\dfrac{Cx}{x(x+1)^2}=\dfrac{A(x+1)^2+Bx(x+1)+Cx}{x(x+1)^2}$ so

$A(x+1)^2+Bx(x+1)+Cx=5x^2+13x+3$; to solve for A , B and C :

Let $x=-1 \implies A(-1+1)^2+B(-1)(-1+1)+C(-1)=5(-1)^2+13(-1)+3 \implies C=5$

Let $x=0$, $C=5 \implies A(0+1)^2+B(0)(0+1)+(5)(0)=5(0)^2+13(0)+3 \implies A=3$

Let $x=1$, $A=3$, $C=5 \implies (3)(1+1)^2+B(1)(1+1)+(5)(1)=5(1)^2+13(1)+3 \implies B=2$

therefore $\dfrac{5x^2+13x+3}{x(x+1)^2}=\dfrac{3}{x}+\dfrac{2}{x+1}+\dfrac{5}{(x+1)^2}$

Complete The Square Method:

Complete the square of a quadratic expression to get an expression like (u^2+a^2) , (u^2-a^2) , (a^2-u^2)

Given ax^2+bx+c , rewrite as $a\left(x^2+\dfrac{b}{a}x\right)+c$, then use the coefficient of x to find a number $\left(\dfrac{b}{2a}\right)^2$;
then add and subtract that number inside the parenthesis;
the first 3 terms inside the parenthesis are a perfect square as shown in the following examples:

Example 1: Given $x^2+2x+10$; the number is $\left(\dfrac{2}{2(1)}\right)^2=1$; add and subtract 1 , we get:

$x^2+2x+1-1+10=(x^2+2x+1)+9=(x+1)^2+9$

Therefore we change $x^2+2x+10$ into $(x+1)^2+9$ like u^2+a^2

Example 2: Given $x^2-6x+13$; the number is $\left(\dfrac{-6}{2(1)}\right)^2=9$; add and subtract 9 , we get:

$x^2-6x+9-9+13=(x^2-6x+9)+4=(x-3)^2+4$

Therefore we change $x^2-6x+13$ into $(x-3)^2+4$ like u^2+a^2

Complete The Square Method:

Example 3: Given $2x^2 - 4x + 3$; rewrite as $2(x^2 - 2x) + 3$, the number is $\left(\frac{-2}{2(1)}\right)^2 = 1$; add and subtract 1 inside the parenthesis , we get:

$2(x^2 - 2x + 1 - 1) + 3 = 2(x^2 - 2x + 1) - 2 + 3 = 2(x-1)^2 + 1$

Therefore we change $2x^2 - 4x + 3$ into $2(x-1)^2 + 1$

Example 4: Given $8 + 4x - x^2$; rewrite as $8 - (x^2 - 4x)$, the number is $\left(\frac{-4}{2(1)}\right)^2 = 4$; add and subtract 4 inside the parenthesis , we get:

$8 - (x^2 - 4x + 4 - 4) = 8 + 4 - (x^2 - 4x + 4) = 12 - (x-2)^2$

Therefore we change $8 + 4x - x^2$ into $12 - (x-2)^2$ like $a^2 + u^2$

Example 5: Given $x^2 + 8x + 15$; rewrite as $(x^2 + 8x) + 15$, the number is $\left(\frac{8}{2(1)}\right)^2 = 16$; add and subtract 16 inside the parenthesis , we get:

$(x^2 + 8x + 16 - 16) + 15 = (x^2 + 8x + 16) - 1 = (x+4)^2 - 1$

Therefore we change $x^2 + 8x + 15$ into $(x+4)^2 - 1$ like $u^2 - a^2$