Definition: Given a function $f(x)$ as a graph or an equation, find x value c such that $f(x)$ is continuous.
A function $f(x)$ at $x = c$ is continuous if 3 conditions are verified.
If one of the conditions fail, we say that $f(x)$ is discontinuous at $x = c$.
The 3 conditions are:

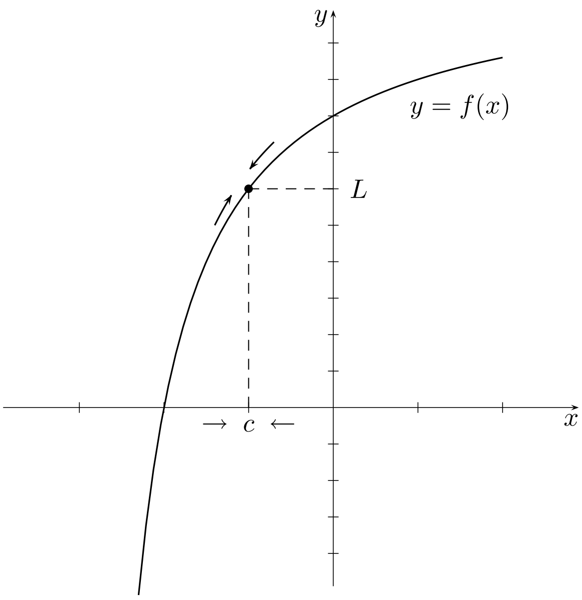
(a) point exists at $x = c\,,\, y = f(c)$. Full dot on the graph of $f(x)$.
An empty dot indicates that the point does not exist (D.N.E.) and condition (a) fails.

(b) the limit of the function $f(x)$ at $x = c$ exists.
The limit on both sides of $x = c$ has the same value $L \longrightarrow \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$
if the limit on both sides is different, then the limit D.N.E. and the condition (b) fails.

(c) $\lim_{x\to c} f(x) = L = f(c)$. If not equal, condition (c) fails.

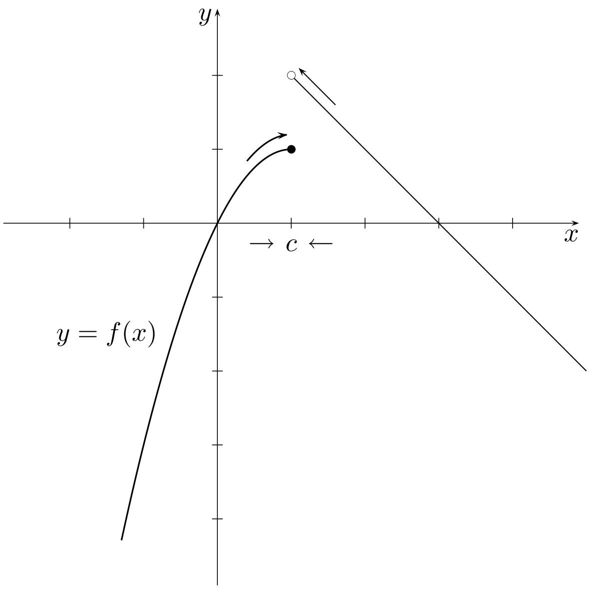
Graphical Continuity:

continuous at $x = c$



All 3 conditions are verified

discontinuous at $x = c$



conditions (b) and (c) fail

Continuity notation and symbol:

A function $f(x)$ is continuous at $x = c$ on a graph or an equation if the 3 conditions are verified.
If one condition fails, then $f(x)$ is discontinuous at $x = c$

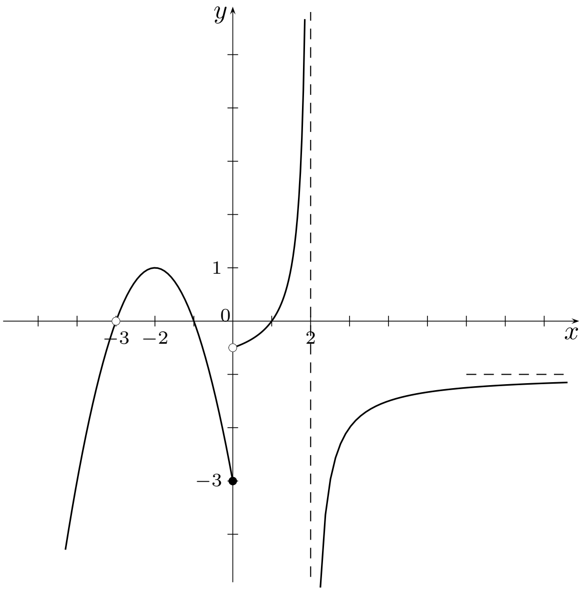
(a) point exists at $x = c\,,\, y = f(c)$

(b) limit exists if $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) \implies \lim_{x\to c} f(x) = L$

(c) $\lim_{x\to c} f(x) = L = f(c)$

Note: Pay attention to the presentation of the notation.

Example 1: Given the graph of a function $f(x)$,
use the definition of continuity to verify the continuity of $f(x)$ at:
(a) $x = -3$; (b) $x = -2$; (c) $x = 0$; (d) $x = 2$



(1 a) condition (a) fails: point D.N.E.
(empty dot).
condition (b): limit exists→ $\lim_{x\to -3} f(x) = 0$
condition (c) fails: $\lim_{x\to -3} f(x) \neq f(-3)$
since conditions (a) and (c) failed,
then $f(x)$ is discontinuous at $x = -3$.

(1 b) condition (a): point exists at $(-2, 1)$
condition (b): limit exists→ $\lim_{x\to -2} f(x) = 1$
condition (c): $\lim_{x\to -2} f(x) = f(-2) = 1$
all 3 conditions are verified,
then $f(x)$ is continuous at $x = -2$.

(1 c) condition (a): point exists at $(0, -3)$
condition (b) fails: limit D.N.E.
$\lim_{x\to 0^-} f(x) = -3$; $\lim_{x\to 0^+} f(x) = -\frac{1}{2}$
condition (c) fails: $\lim_{x\to 0} f(x) \neq f(0)$
since conditions (b) and (c) failed,
then $f(x)$ is discontinuous at $x = 0$.

(1 d) condition (a) fails: point D.N.E. at $x = 2$ (vertical asymptote)
condition (b) fails: limit D.N.E. → $\lim_{x\to 2^-} f(x) = +\infty$; $\lim_{x\to 2^+} f(x) = -\infty$
condition (c) fails: $\lim_{x\to 2} f(x) \neq f(2)$
since all 3 conditions failed, then $f(x)$ is discontinuous at $x = 2$.

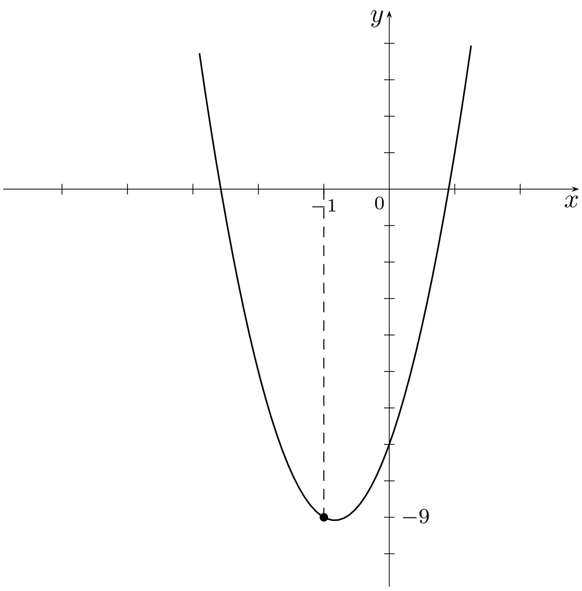
Example 2: Given the equation of a function $y = f(x) = 3x^2 + 5x - 7$, test for continuity at $x = -1$.

Condition (a): point exists at
$x = -1$, $y = f(-1) = 3(-1)2 + 5(-1) - 7 = -9$

Condition (b): limit exists since
$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x)$
$= \lim_{x\to -1} (3x^2 + 5x - 7) = -9$

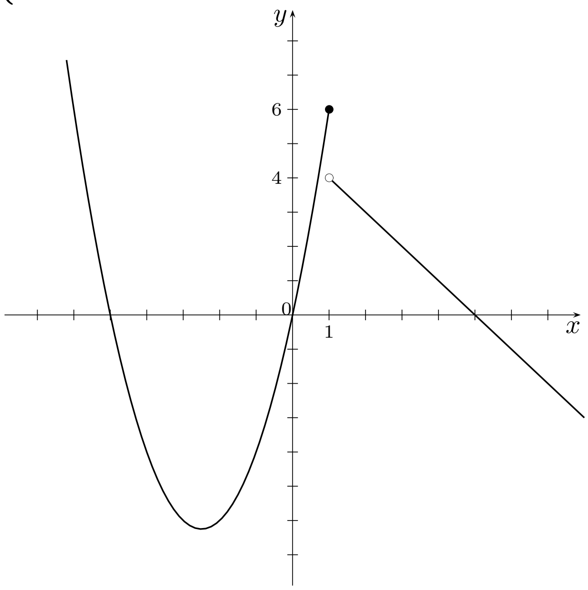
Condition (c): $\lim_{x\to -1} f(x) = f(-1) = -9$

Since all 3 conditions are verified,
then $f(x)$ is continuous at $x = -1$.



Note: The graph is not part of the solution.
It is just visual !

Example 3: Given the equation of a function $y = f(x) = \begin{cases} x^2 + 5x & \text{if} \quad x \leq 1 \\ -x + 4 & \text{if} \quad x > 1 \end{cases}$, test for continuity at $x = 1$.



$f(x)$ is a piecewise :
at $x < 1$ (left side of $x = 1$)
use the upper equation;
at $x > 1$ (right side of $x = 1$)
use the lower equation.

Condition (a): point exists at $x = 1$, $y = f(1) = 6$

Condition (b) fails: limit D.N.E.
$\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} (x^2 + 5x) = 6$
$\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} (-x + 5) = 4$

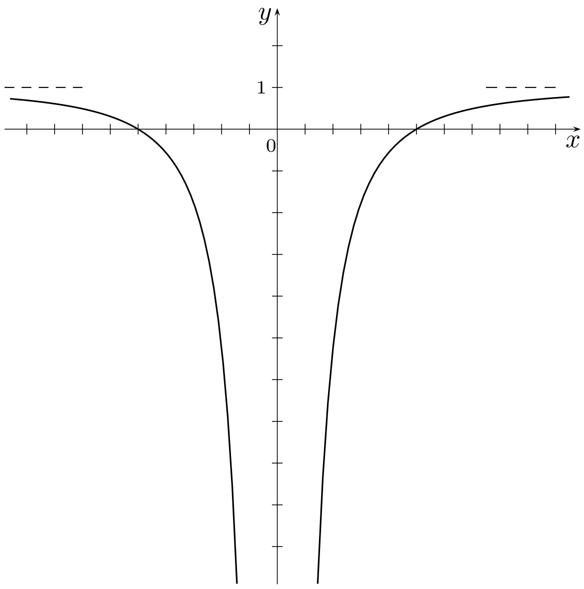
Condition (c) fails: $\lim\limits_{x\to 1} f(x) \neq f(1)$

Since conditions (b) and (c) failed,
then $f(x)$ is discontinuous at $x = 1$.

Note:
Pay attention to the presentation of the notation.

Note: The graph is not part of the solution.
It is just visual !

Example 4: Given the equation of a function $y = f(x) = \dfrac{x^2 - 25}{x^2}$, for what x value(s) $f(x)$ is discontinuous?



$f(x)$ is continuous for x real numbers except
when denominator is 0.
The denominator of $f(x)$ is 0 at $x = 0$.
Test the conditions of continuity at $x = 0$.

Condition (a) fails: point D.N.E.
since at $x = 0$, $f(0)$ is undefined.

Condition (b): limit exists since
$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} f(x) = -\infty$

Condition (c) fails: $\lim\limits_{x\to 0} f(x) \neq f(0)$

Since conditions (a) and (c) failed,
then $f(x)$ is discontinuous at $x = 0$.

Note:
Pay attention to the presentation of the notation.

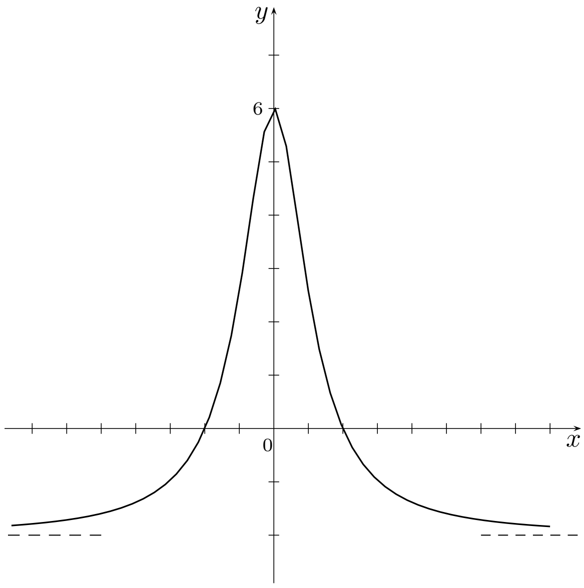
Note: The graph is not part of the solution.
It is just visual !

Example 5: Given the equation of a function $y = f(x) = \dfrac{24 - 6x^2}{3x^2 + 4}$, for what x value(s) $f(x)$ is continuous?

$f(x)$ is continuous for all x real numbers since the denominator can never equal 0.

All 3 conditions for continuity are verified for every x value in $f(x)$.

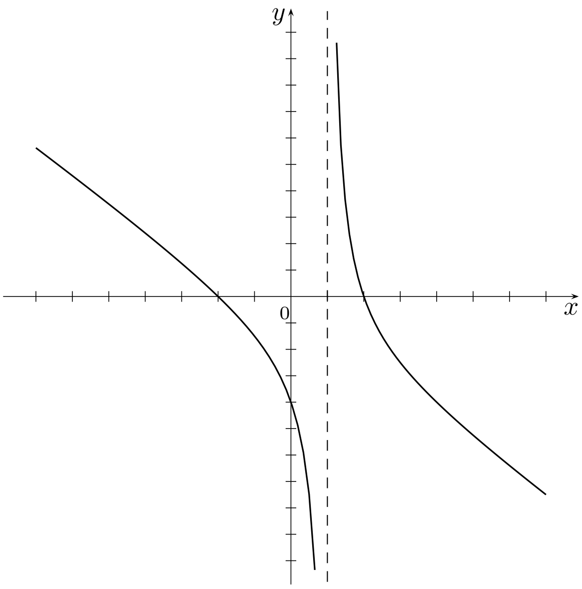
Note:
in any rational function , there is a solution for x if the denominator equals 0 , these x value(s) are the location where $f(x)$ is discontinuous since at these x value(s), the point(s) D.N.E. ; there is no solution if the denominator cannot equal 0 then $f(x)$ is continuous for all x .



Note: The graph is not part of the solution.
It is just visual !

Example 6: Given the equation of a function $y = f(x) = \dfrac{4 - x^2}{x - 1}$, test the continuity of $f(x)$ at:
(a) $x = -2$; (b) $x = 1$; (c) $x = 3$

Note: The solution of each question must be shown without using a graph.
At the end of all 3 solutions , a graph of $f(x)$ is displayed for a visual verification.



(6 a) condition (a): point exists at
$x = -2$, $y = f(-2) = 0$
condition (b): limit exists since
$\lim_{x \to -2} f(x) = 0$
condition (c): $\lim_{x \to -2} f(x) = f(-2) = 0$
since all 3 conditions are verified,
then $f(x)$ is continuous at $x = -2$.
(6 b) condition (a) fail: point D.N.E. at $x = 1$,(vertical asymptote)
condition (b) fail: limit D.N.E. since
$\lim_{x \to 1^-} f(x) = -\infty$; $\lim_{x \to 1^+} f(x) = +\infty$
condition (c) fails: $\lim_{x \to 1} f(x) \neq f(1)$
all 3 conditions failed,
then $f(x)$ is discontinuous at $x = 1$.
When the denominator equals 0,
the solution is $x = 1$
and $f(x)$ is discontinuous at this value.
(6 c) condition (a): point exists at
$x = 3$, $y = f(3) = -\frac{5}{2}$
condition (b): limit exists since
$\lim_{x \to 3} f(x) = -\frac{5}{2}$
condition (c): $\lim_{x \to 3} f(x) = f(3) = \frac{5}{2}$
since all 3 conditions are verified,
then $f(x)$ is continuous at $x = 3$.

Example 7: Given the equation of a function $y = f(x) = \dfrac{x^2 - 2x - 3}{x^2 - 1}$, test the continuity of $f(x)$ at $x = -1$.

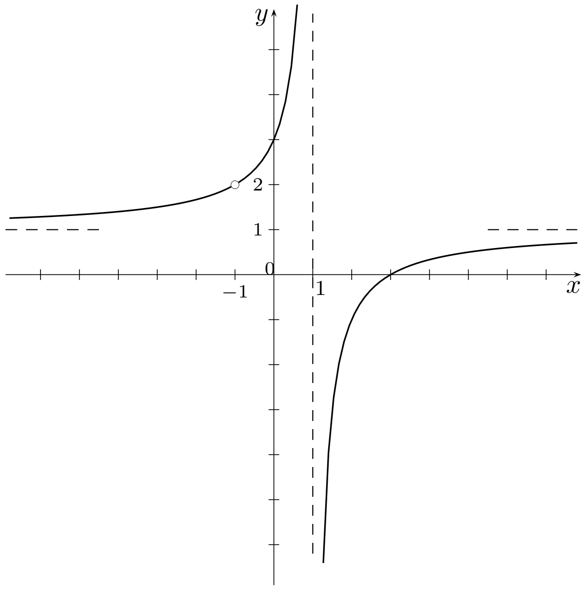
condition (a) fails: point D.N.E. since
$x = -1$, $y = f(-1) = \frac{0}{0}$
indeterminate; empty dot.

condition (b): limit exists after factoring and reducing:
$\dfrac{(x+1)(x-3)}{(x+1)(x-1)} = \dfrac{x-3}{x-1} \implies \lim\limits_{x \to -1} \dfrac{x-3}{x-1} = 2$

condition (c) fails: $\lim\limits_{x \to -1} f(x) \neq f(-1)$

conditions (a) and (c) failed,
then $f(x)$ is discontinuous at $x = -1$.

Note: indetermine form implies Algebra work to be done to remove the indetermination. Indetermine (empty dot) is not the same as undefined (vertical asymptote)!



the curve has an empty dot at point $(-1, 2)$.
The empty dot indicates a break in the curve.
Note: The graph is not part of the solution.
It is just visual !

Example 8: Given the equation of a function $y = f(x) = \dfrac{\frac{4}{x-2} - 2}{x - 4}$, test the continuity of $f(x)$ at $x = 4$.

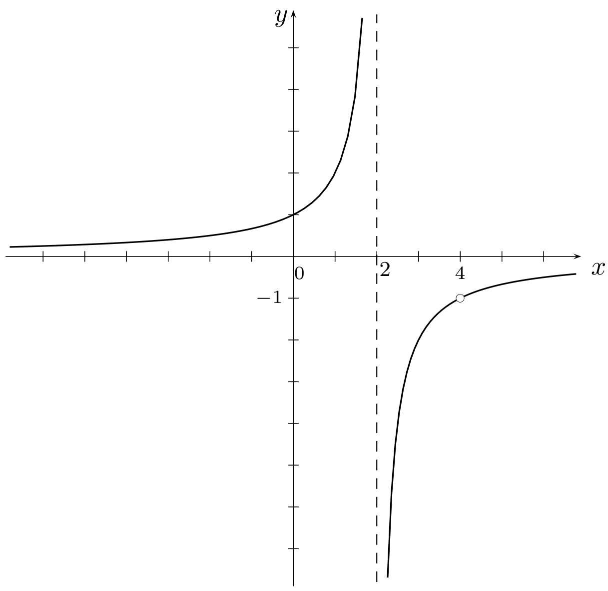
condition (a) fails: point D.N.E. since
$x = 4$, $y = f(4) = \frac{0}{0}$
indeterminate; empty dot.

condition (b): limit exists after
common denominator and reducing:
$\dfrac{\frac{4-2(x-2)}{x-2}}{x-4} = \dfrac{\frac{8-2x}{x-2}}{x-4} = \dfrac{-2(x-4)}{x-2} \cdot \dfrac{1}{x-4}$
$= \dfrac{-2}{x-2} \implies \lim\limits_{x \to 4} \dfrac{-2}{x-2} = -1$

condition (c) fails: $\lim\limits_{x \to 4} f(x) \neq f(4)$

conditions (a) and (c) failed,
then $f(x)$ is discontinuous at $x = 4$.

Note: Algebra work to be done to remove the indetermination is the common denominator to reduce fractions.



the curve has an empty dot at point $(4, -1)$.
The empty dot indicates a break in the curve.
Note: The graph is not part of the solution.
It is just visual !

Example 9: Given the equation of a function $y = f(x) = \dfrac{\sqrt{x+6}-3}{x-3}$, test the continuity of $f(x)$ at $x = 3$.

condition (a) fails: point D.N.E. since
$x = 3\,,\ y = f(3) = \frac{0}{0}$
indeterminate; empty dot.

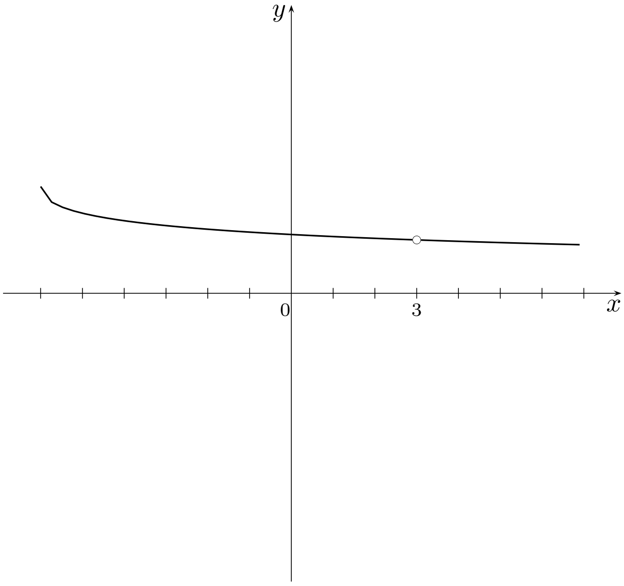
condition (b): limit exists after
rationalize numerator and reducing:

$$\frac{(\sqrt{x+6}-3)(\sqrt{x+6}+3)}{(x-3)(\sqrt{x+6}+3)}$$
$$= \frac{x+6-9}{(x-3)(\sqrt{x+6}+3)} = \frac{x-3}{(x-3)(\sqrt{x+6}+3)}$$
$$\lim_{x\to 3} \frac{1}{\sqrt{x+6}+3} = \frac{1}{6}$$

condition (c) fails: $\lim\limits_{x\to 3} f(x) \neq f(3)$

conditions (a) and (c) failed,
then $f(x)$ is discontinuous at $x = 3$.

Note: Algebra work to be done to remove the indetermination is rationalize numerator to reduce radicals.



the curve has an empty dot at point $\left(3, \frac{1}{6}\right)$.
The empty dot indicates a break in the curve.
Note: The graph is not part of the solution.
It is just visual !