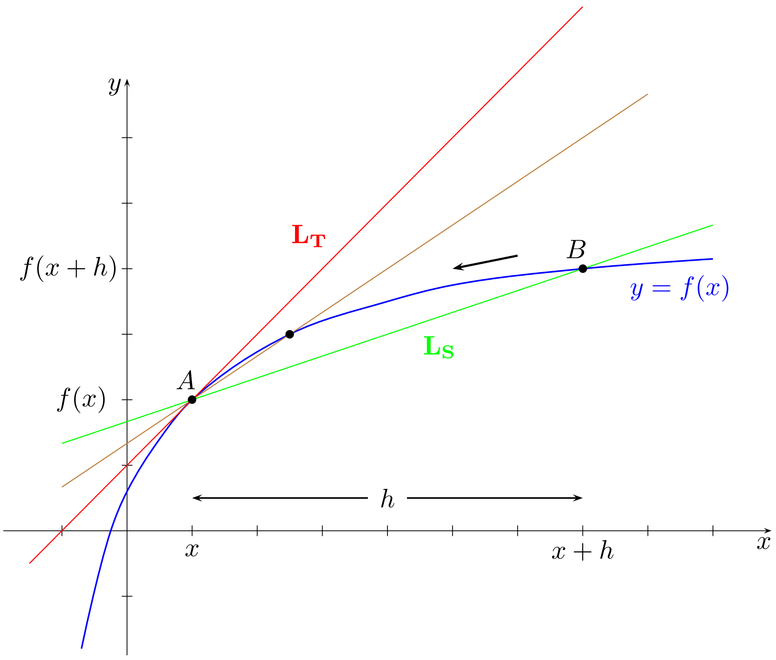
Graphical Definition of the Derivative



Examine point A $\left(x, f(x)\right)$ and point B $\left(x+h, f(x+h)\right)$

To find the Slope of tangent line:

Let the segment joining points A and B is called a secant line L_S (green line).

Slope of the secant line through points A and B: $m_{L_S} = \dfrac{\Delta y}{\Delta x} = \dfrac{f(x+h) - f(x)}{x+h-x} = \dfrac{f(x+h) - f(x)}{h}$

As point B moves on the curve of $f(x)$ towards point A, the secant line L_S change position as shown on the previous graph. The slope of the secant line L_S also changes. As point B gets close to point A on the curve of $f(x)$, the distance between the x-values of both points $(x+h) - x = h$ is getting smaller. As point B approaches point A, value of h approaches 0 and the slope of the secant line L_S approaches the value of the slope of the tangent line L_T at point A (red line). In mathematical terms, the limit concept is used:

$$m_{\mathbf{L_S}} = m_{\mathbf{L_T}} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Derivative of a function:

The slope of the tangent line L_T at point A with coordinates $\left(x, f(x)\right)$ represents the derivative of the function $f(x)$ at any point on the curve of $f(x)$.

The notation of the derivative of a function $y = f(x)$ is

$$\boxed{y' = f'(x) = \frac{dy}{dx} = \frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

This is the definition of the derivative of function $y = f(x)$

Derivative of a function:

In the following examples, you will apply this definition to find the derivative of a function $y = f(x)$. To find the slope of the tangent line at a specific x value a, replace this value in the derivative $y' = f'(x)$ and you will get a value $y'(a) = f'(a)$ representing the slope of the tangent line at that position on the curve.

Example 1: Given the function $f(x) = 3x^2$, use the definition of the derivative to find the derivative of $f(x)$.

The definition of the derivative is $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

We need to replace $f(x)$ with $3x^2$ and $f(x+h)$ with $3(x+h)^2$

Notice that $f(x+h)$ is a composite operation requiring to replace each x in $f(x)$ with $(x+h)$.

Next we determine the derivative of $f(x)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{3(x+h)^2 - 3x^2}{h} = \frac{0}{0}$$

Note: Algebra work is needed to remove the indetermination.

$$f'(x) = \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 3x^2}{h} = \lim_{h \to 0} \frac{6xh + 3h^2}{h} = \lim_{h \to 0} \frac{h(6x + 3h)}{h} = \lim_{h \to 0} (6x + 3h) = 6x$$

(a) Evaluate the slope of the tangent line at $x = 1$.
The derivative of $f(x)$ is $f'(x) = 6x$
replace in the derivative x with 1 to find the slope of the tangent line at $x = 1$

$$m = f'(1) = \left.\frac{df}{dx}\right|_{x=1} = 6(1) = 6$$

Note: Pay attention to different notations representing slope of tangent line at $x = 1$

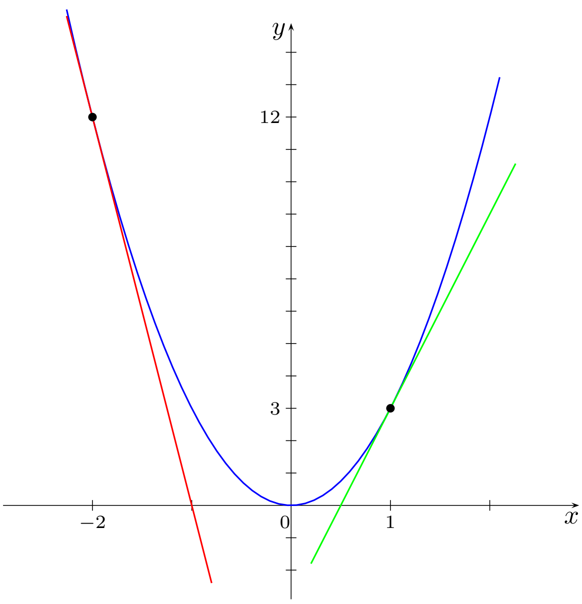
(b) Find the equation of the tangent line at $x = -2$.
The derivative of $f(x)$ is $f'(x) = 6x$
replace in the derivative x with -2 to find the
slope of the tangent line at $x = -2$

$$m = f'(-2) = \left.\frac{df}{dx}\right|_{x=-2} = 6(-2) = -12$$

Equation of tangent line: $y = mx + b$
need to replace slope $m = f'(-2) = -12$ and point at $x = -2$, $y = f(-2) = 12$ to solve b
$12 = -12(-2) + b \implies b = -12$

$y = -12x - 12$

The graph shows the curve of $f(x)$ (blue curve) and the tangent lines at $x = -2$ (red line) and $x = 1$ (green line)



Example 2: Given the function $f(x) = \dfrac{5}{x} = 5x^{-1}$, use the definition of the derivative to find the derivative of $f(x)$.

The definition of the derivative is $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

We need to replace $f(x)$ with $\dfrac{5}{x}$ and $f(x+h)$ with $\dfrac{5}{x+h}$

Notice that $f(x+h)$ is a composite operation requiring to replace each x in $f(x)$ with $(x+h)$.

Next we determine the derivative of $f(x)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{5}{x+h} - \frac{5}{x}}{h} = \frac{0}{0}$$

<u>Note</u>: Algebra work is needed to remove the indetermination.

$$f'(x) = \lim_{h\to 0} \frac{\frac{5x-5(x+h)}{x(x+h)}}{h} = \lim_{h\to 0} \frac{-5h}{x(x+h)} \cdot \frac{1}{h} = \lim_{h\to 0} \frac{-5}{x(x+h)} = \frac{-5}{x^2} = -5x^{-2}$$

(a) Evaluate the slope of the tangent line at $x = 1$.
The derivative of $f(x)$ is $f'(x) = -\dfrac{5}{x^2}$
replace in the derivative x with 1 to find the slope of the tangent line at $x = 1$

$$m = f'(1) = \left.\frac{df}{dx}\right|_{x=1} = -\frac{5}{(1)^2} = -5$$

<u>Note</u>: Pay attention to different notations representing slope of tangent line at $x = 1$

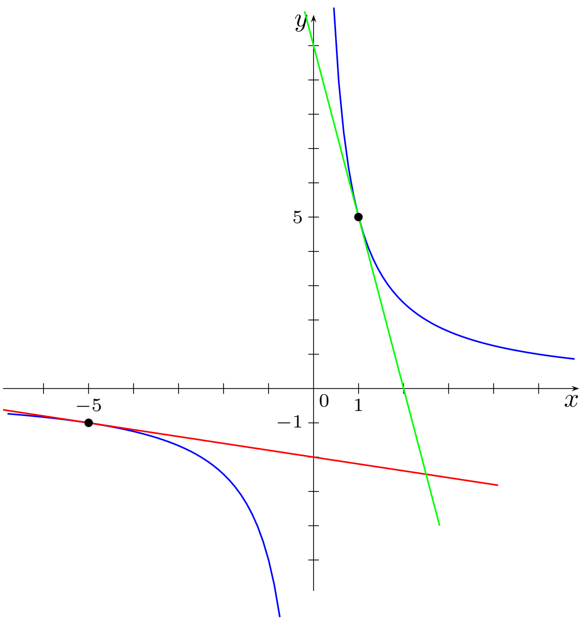
(b) Find the equation of the tangent line at $x = -5$.
The derivative of $f(x)$ is $f'(x) = -\dfrac{5}{x^2}$
replace in the derivative x with -5 to find the slope of the tangent line at $x = -5$

$$m = f'(-5) = \left.\frac{df}{dx}\right|_{x=-5} = -\frac{5}{(-5)^2} = -\frac{1}{5}$$

Equation of tangent line: $y = mx + b$
need to replace slope $m = f'(-5) = -\dfrac{1}{5}$ and point at $x = -5$, $y = f(-5) = -1$ to solve b
$-1 = -\frac{1}{5}(-1) + b \implies b = -2$

$y = -\frac{1}{5}x - 2$

The graph shows the curve of $f(x)$ (blue curve) and the tangent lines at $x = -5$ (red line) and $x = 1$ (green line)



Example 3: Given the function $f(x) = 4\sqrt{x} = 4x^{1/2}$, use the definition of the derivative to find the derivative of $f(x)$.

The definition of the derivative is $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

We need to replace $f(x)$ with $4\sqrt{x}$ and $f(x+h)$ with $4\sqrt{x+h}$

Notice that $f(x+h)$ is a composite operation requiring to replace each x in $f(x)$ with $(x+h)$.

Next we determine the derivative of $f(x)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{4\sqrt{x+h}-4\sqrt{x}}{h} = \frac{0}{0}$$

Note: Algebra work is needed to remove the indetermination.

$$f'(x) = \lim_{h\to 0} \frac{\left(4\sqrt{x+h}-4\sqrt{x}\right)\left(4\sqrt{x+h}+4\sqrt{x}\right)}{h\left(4\sqrt{x+h}+4\sqrt{x}\right)} = \lim_{h\to 0} \frac{16(x+h)-16x}{h\left(4\sqrt{x+h}+4\sqrt{x}\right)}$$

$$f'(x) = \lim_{h\to 0} \frac{16h}{h\left(4\sqrt{x+h}+4\sqrt{x}\right)} = \frac{16}{4\sqrt{x+h}+4\sqrt{x}} = \frac{16}{8\sqrt{x}} = \frac{2}{\sqrt{x}} = 2x^{-1/2}$$

(a) Evaluate the slope of the tangent line at $x = 1$.
The derivative of $f(x)$ is $f'(x) = \dfrac{2}{\sqrt{x}}$
replace in the derivative x with 1 to find the slope of the tangent line at $x = 1$

$$m = f'(1) = \left.\frac{df}{dx}\right|_{x=1} = \frac{2}{\sqrt{1}} = 2$$

Note: Pay attention to different notations representing slope of tangent line at $x = 1$

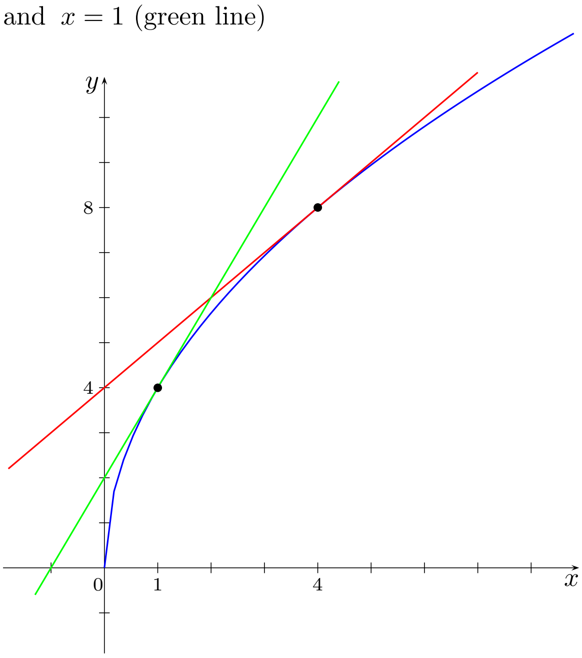
(b) Find the equation of the tangent line at $x = 4$.
The derivative of $f(x)$ is $f'(x) = \dfrac{2}{\sqrt{x}}$
replace in the derivative x with 4 to find the slope of the tangent line at $x = 4$

$$m = f'(4) = \left.\frac{df}{dx}\right|_{x=4} = \frac{2}{\sqrt{4}} = 1$$

Equation of tangent line: $y = mx + b$
need to replace slope $m = f'(4) = 1$ and point at $x = 4$, $y = f(4) = 8$ to solve b
$8 = 1(4) + b \implies b = 4$

$y = x + 4$

The graph shows the curve of $f(x)$ (blue curve) and the tangent lines at $x = 4$ (red line) and $x = 1$ (green line)



Example 4: Given the function $f(x) = 6$, use the definition of the derivative to find the derivative of $f(x)$.

The definition of the derivative is $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

We need to replace $f(x)$ with 6 and $f(x+h)$ with 6

Notice that $f(x+h)$ is a composite operation requiring to replace each x in $f(x)$ with $(x+h)$;
since no x is shown in $f(x)$ =constant, then $f(x+h)$ remains the same.

Next we determine the derivative of $f(x)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{6-6}{h} = \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} (0) = 0$$

Note: The derivative of a constant is zero.
The constant function is a horizontal line with slope 0.

Example 5: Given the function $f(x) = x^2 - 2x$, use the definition of the derivative to find the derivative of $f(x)$.

The definition of the derivative is $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

We need to replace $f(x)$ with $x^2 - 2x$ and $f(x+h)$ with $(x+h)^2 - 2(x+h)$

Notice that $f(x+h)$ is a composite operation requiring to replace each x in $f(x)$ with $(x+h)$;

Next we determine the derivative of $f(x)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\left[(x+h)^2 - 2(x+h)\right] - \left[x^2 - 2x\right]}{h} = \frac{0}{0}$$

Note: Algebra work is needed to remove the indetermination.

$$f'(x) = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 2x - 2h - x^2 + 2x}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{2xh + h^2 - 2h}{h} = \lim_{h \to 0} \frac{h(2x + h - 2)}{h} = \lim_{h \to 0} (2x + h - 2) = 2x - 2$$

Example 5:

(a) Evaluate the slope of the tangent line at $x = 2$.
The derivative of $f(x)$ is $f'(x) = 2x - 2$ replace in the derivative x with 2 to find the slope of the tangent line at $x = 2$

$$m = f'(2) = \left.\frac{df}{dx}\right|_{x=2} = 2(2) - 2 = 2$$

Note: Pay attention to different notations representing slope of tangent line at $x = 2$

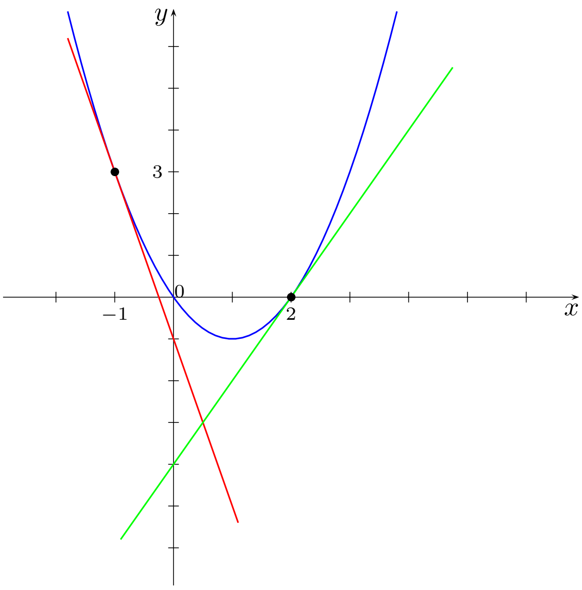
(b) Find the equation of the tangent line at $x = -1$.
The derivative of $f(x)$ is $f'(x) = 2x - 2$ replace in the derivative x with -1 to find the slope of the tangent line at $x = -1$

$$m = f'(-1) = \left.\frac{df}{dx}\right|_{x=-1} = 2(-1) - 2 = -4$$

Equation of tangent line: $y = mx + b$
need to replace slope $m = f'(-1) = -4$ and point at $x = -1$, $y = f(-1) = 3$ to solve b
$3 = -4(-1) + b \implies b = -1$

$y = -4x - 1$

The graph shows the curve of $f(x)$ (blue curve) and the tangent lines at $x = -1$ (red line) and $x = 2$ (green line)



Differentiability:

Definition:
A function $f(x)$ is differentiable at $x = a$ if $f(x)$ is continuous at $x = a$ and $f'(a)_{-} = f'(a)_{+}$

(a) the curve of $f(x)$ has no break or no vertical asymptote at $x = a$ (no discontinuity)

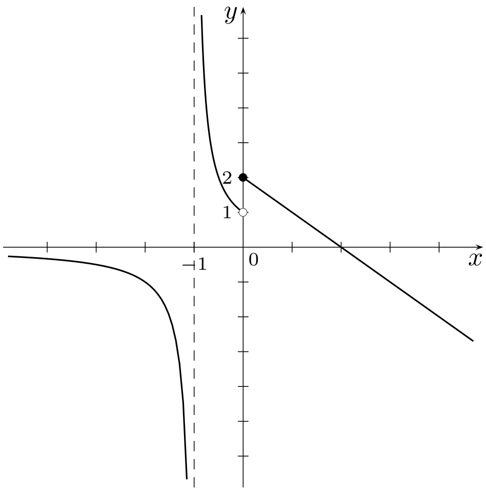
(b) no vertical tangent at $x = a$ since the derivative is undefined,

(c) no sharp turn on the curve of $f(x)$ at $x = a$ since the slope of the tangent at the left of $x = a$ will not be equal to the slope of the tangent at the right of $x = a$

A function $f(x)$ is not differentiable at $x = a$ if $f(x)$ is discontinuous at $x = a$ and/or $f'(a)_{-} \neq f'(a)_{+}$

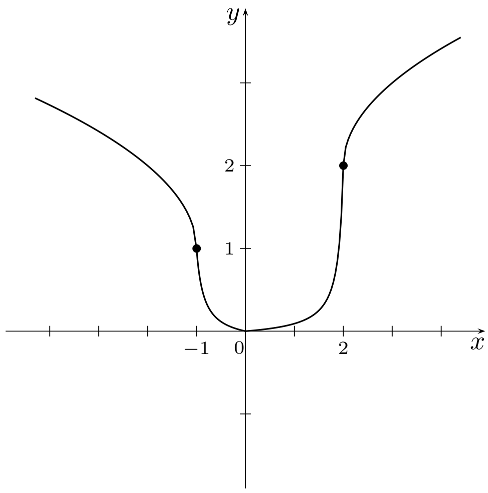
The following graphs show that the curve of $f(x)$ is not differentiable at $x = a$

The curve of $f(x)$ has a break at $x = a$ (discontinuity)



The function $f(x)$ is nondifferentiable at $x = -1$ (vertical asymptote) and at $x = 0$ (curve breaks)

$f(x)$ has a vertical tangent at $x = a$, therefore the derivative (slope of tangent line) is undefined



The function $f(x)$ is nondifferentiable at $x = -1$ and $x = 2$ (vertical tangent line to the curve at these points)

A sharp turn on the curve of $f(x)$ at $x = a$ shows the slope of the tangent at the left of $x = a$ is not equal to the slope of the tangent at the right of $x = a$

curve of $f(x)$ $\longrightarrow$

The function $f(x)$ is nondifferentiable at $x = -1$, $x = 2$ and $x = 3$ (sharp turn at these points)