Critical numbers - Increasing and decreasing intervals - Relative Extrema

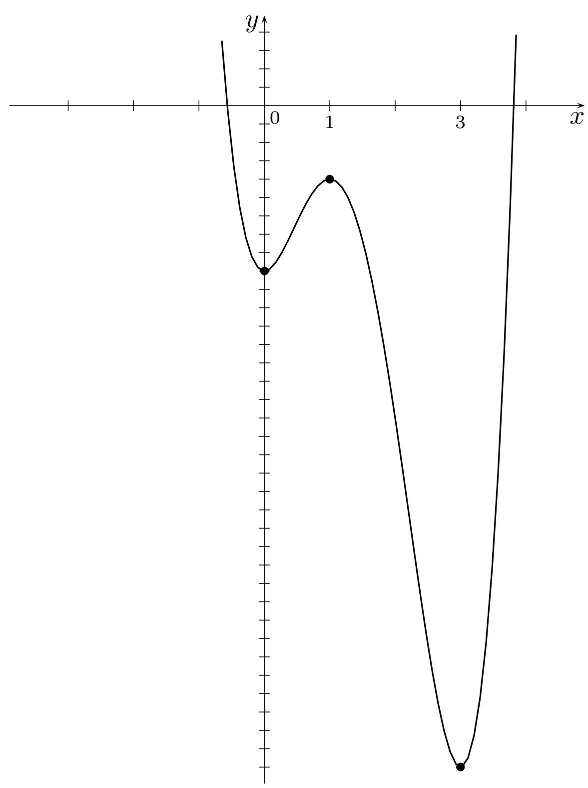
Given $f(x)$, the derivatives $f'(x)$ and $f''(x)$ give important information about $f(x)$.
$f'(x)$ will help find the critical numbers (C.N.).
To find them, make the answer of $f'(x)$ equal to zero; finally solve for x and these value(s) are called C.N. of $f(x)$.
Use these C.N. for the first derivative test to find the intervals where $f(x)$ is increasing or decreasing and the relative extrema.

Example 1: Given $f(x) = 3x^4 - 16x^3 + 18x^2 - 9$, find:

(a) the critical numbers of $f(x)$
$f'(x) = 12x^3 - 48x^2 + 36x = 12x(x-1)(x-3)$
$f'(x) = 0 \implies 12x(x-1)(x-3) = 0$
the critical numbers are $x = 0$, $x = 1$ and $x = 3$

(b) the intervals where $f(x)$ is increasing or decreasing.
Use the first derivative test by locating C.N. on the number line and test $f'(x)$ for each interval.
If $f'(x) > 0$, $f(x)$ is increasing; $f'(x) < 0$, $f(x)$ is decreasing.

(c) the relative extrema of $f(x)$
To find the extrema, examine each C.N. in the first derivative test to decide:
If before the C.N. $f'(x) > 0$ and after the C.N. $f'(x) < 0$, then this C.N. is a relative maximum and find the corresponding y value.
If before the C.N. $f'(x) < 0$ and after the C.N. $f'(x) > 0$, then this C.N. is a relative minimum and find the corresponding y value.
From the first derivative test, we get at $x = 0$, a relative minimum and the y value is $f(0) = -9$
at $x = 1$, a relative maximum and $f(1) = -4$ and
at $x = 3$, a relative minimum with $f(3) - 36$



First Derivative Test

$-\quad +\quad -\quad +$

$\searrow 0 \nearrow 1 \searrow 3 \nearrow$

Note:
$f(x)$ is a polynomial function without asymptotes.

$f(x)$ is increasing at
$0 < x < 1$ or $x > 3$
$f(x)$ is decreasing at
$x < 0$ or $1 < x < 3$

<u>Relative extrema</u> are:
relative minimum at
$(0, -9)$ and$(3, -36)$
relative maximum at
$(1, -4)$

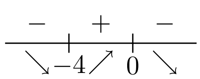
Example 2: Given $f(x) = \dfrac{x+2}{2x^2}$, find:

(a) the critical numbers of $f(x)$
$f'(x) = -\dfrac{x+4}{2x^3}$: numerator= 0 $\implies x+4=0 \implies x=-4$; denominator= 0 $\implies 2x^3=0 \implies x=0$
the critical numbers are $x=-4$ and $x=0$

(b) the intervals where $f(x)$ is increasing or decreasing.
Use the first derivative test by locating C.N. on the number line and test $f'(x)$ for each interval.
If $f'(x) > 0$, $f(x)$ is increasing; $f'(x) < 0$, $f(x)$ is decreasing.

(c) the relative extrema of $f(x)$
To find the extrema, examine each C.N. in the first derivative test to decide:
If before the C.N. $f'(x) > 0$ and after the C.N. $f'(x) < 0$, then this C.N. is a relative maximum and find the corresponding y value.
If before the C.N. $f'(x) < 0$ and after the C.N. $f'(x) > 0$, then this C.N. is a relative minimum and find the corresponding y value.
From the first derivative test, we get at $x=0$, no extrema since the y value is undefined,
a vertical asymptote at $x=0$; at
$x=-4$, a relative minimum and $f(-4) = -\frac{1}{16} = -0.0625$

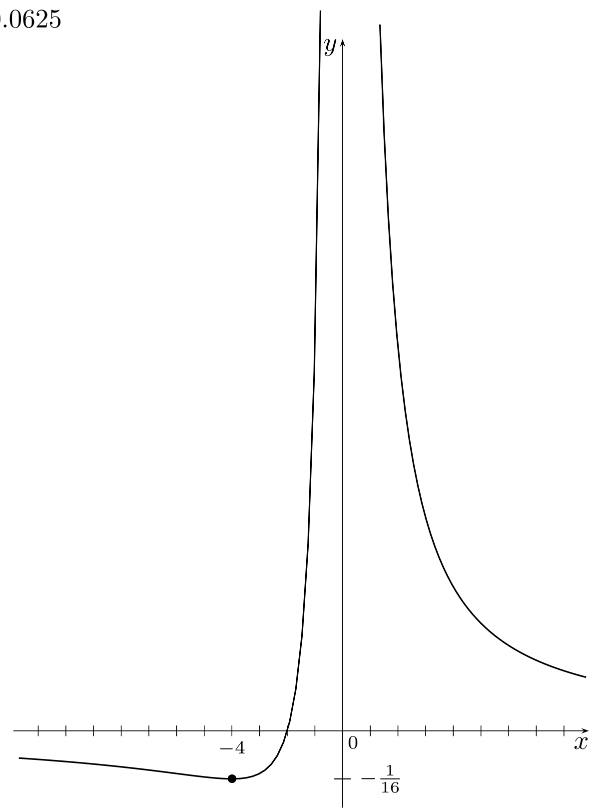
First Derivative Test



$f(x)$ is increasing at
$-4 < x < 0$
$f(x)$ is decreasing at
$x < -4$ or $x > 0$

Note:
$f(x)$ is a rational function with asymptotes.

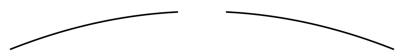
Relative extrema is:
relative minimum at $\left(-4, -\frac{1}{16}\right)$



Concavity: Point(s) of inflection

Given $f(x)$, the derivatives $f'(x)$ and $f''(x)$ give important information about $f(x)$.
$f''(x)$ will help to find the possible point(s) of inflection (P.I.).
To find them, make the answer of $f''(x)$ equal to zero;
finally solve for x and these value(s) are called possible x values P.I. of $f(x)$.
Use these possible P.I. for the second derivative test to find the intervals of concavity and P.I.
Two types of concavity (curve shape):

concave down



$f''(x) < 0$

concave up

$f''(x) > 0$

Example 3: Given $f(x) = x^3 - 27x$, find:

(a) the x values for possible point(s) of inflection of $f(x)$
$f'(x) = 3x^2 - 27$; $f''(x) = 6x$
$f''(x) = 0 \implies 6x = 0$ the x value for possible P.I. is $x = 0$

(b) the intervals where $f(x)$ is concave up or concave down.
Use the second derivative test by locating x value(s)of possible P.I. on the number line and test $f''(x)$ for each interval. If $f''(x) > 0$, $f(x)$ is concave up; $f''(x) < 0$, $f(x)$ is concave down.

Second Derivative Test

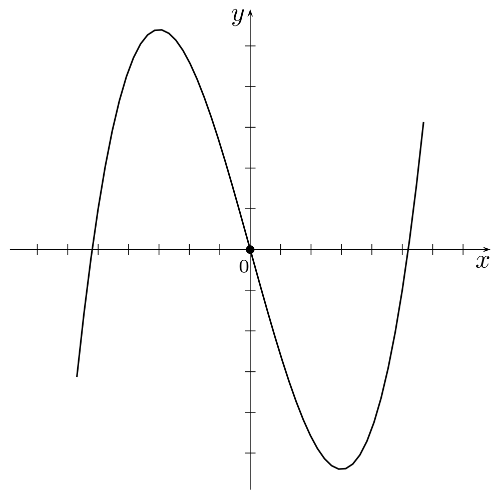
$-$		$+$
$\searrow$	0	$\nearrow$

<u>Note</u>: $f(x)$ is a polynomial function without asymptotes.

(c) any point of inflection for $f(x)$
To find the point(s) of inflection, examine each possible P.I. in the second derivative test to decide:
If before the possible P.I., $f''(x) > 0$ and after the possible P.I., $f''(x) < 0$, then this P.I. is a point of inflection and find the corresponding y value.
If before the possible P.I., $f''(x) < 0$ and after the possible P.I., $f''(x) > 0$, then this P.I. is a point of inflection and find the corresponding y value.
From the second derivative test, we get at
$x = 0$, a point of inflection and the y value is $f(0) = 0$
Point of inflection at $(0, 0)$

$f(x)$ is concave up at $x > 0$
$f(x)$ is concave down at $x < 0$

Point of inflection is: $(0, 0)$



Example 4: Given $f(x) = \dfrac{3x^2}{(x+2)^2}$, find:

(a) the x values for possible point(s) of inflection of $f(x)$

$f'(x) = \dfrac{12x}{(x+2)^3}$; $f''(x) = \dfrac{-24(x-1)}{(x+2)^4}$

from $f''(x)$: numerator= 0 $\Longrightarrow$ $-24(x-1) = 0 \Longrightarrow x = 1$; denominator= 0 $\Longrightarrow (x+2)^4 = 0 \Longrightarrow x = -2$
the x values for possible P.I. are $x = -2$ and $x = 1$

(b) the intervals where $f(x)$ is concave up or concave down.
Use the second derivative test by locating x value(s)of possible P.I. on the number line and test $f''(x)$ for each interval. If $f''(x) > 0$, $f(x)$ is concave up; $f''(x) < 0$, $f(x)$ is concave down.

(c) any point of inflection for $f(x)$
To find the point(s) of inflection, examine each possible P.I. in the second derivative test to decide:
If before the possible P.I., $f''(x) > 0$ and after the possible P.I., $f''(x) < 0$, then this P.I. is a point of inflection and find the corresponding y value.
If before the possible P.I., $f''(x) < 0$ and after the possible P.I., $f''(x) > 0$, then this P.I. is a point of inflection and find the corresponding y value.
From the second derivative test, we get at
$x = -2$, no point of inflection since the y value is undefined,
a vertical asymptote at $x = -2$; at
$x = 1$, a point of inflection and the y value is $f(1) = \frac{1}{3}$
Point of inflection at $\left(1, \frac{1}{3}\right)$

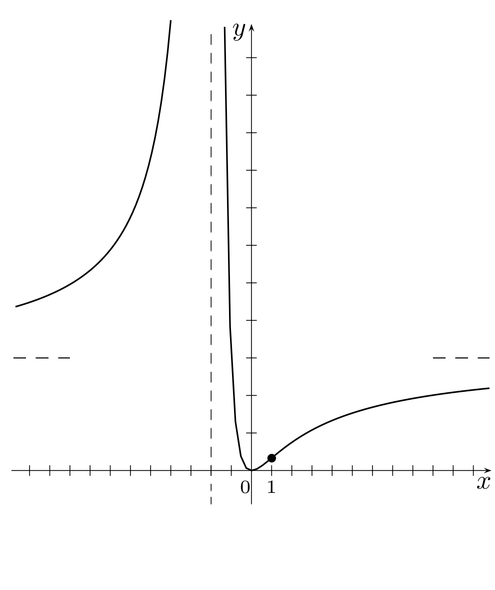
Second Derivative Test

+ | + | −
↗ -2 ↗ 1 ↘

Note:
$f(x)$ is a rational function with asymptotes.

$f(x)$ is concave up at
$x < -2$ or $-2 < x < 1$
$f(x)$ is concave down at
$x > 1$

Point of inflection is: $\left(1, \frac{1}{3}\right)$



Relative Extrema and the second derivative test

Given $f(x)$, we saw that the first derivative test finds the relative extrema and
the second derivatives $f''(x)$ will do the same.
We need $f'(x)$ to find the critical numbers (C.N.). Replace these C.N. in the answer of $f''(x)$.
If $f''(\text{C.N.}) > 0$, then C.N. is a relative minimum and find the corresponding y value.
If $f''(\text{C.N.}) < 0$, then C.N. is a relative maximum and find the corresponding y value.
If $f''(\text{C.N.}) = 0$ or undefined, then C.N. is not a relative extremum.

Example 5: Given $f(x) = 3x^4 - 16x^3 + 18x^2 - 9$, find:
the relative extrema using the second derivative test.

$f'(x) = 12x^3 - 48x^2 + 36x = 12x(x-1)(x-3)$
$f'(x) = 0 \implies 12x(x-1)(x-3) = 0$
the critical numbers are $x = 0$, $x = 1$ and $x = 3$

$f''(x) = 36x^2 - 96x + 36$
replace $x = 0 \implies f''(0) = 36 > 0$, relative minimum at $(0, -9)$
replace $x = 1 \implies f''(1) = -24 < 0$, relative maximum at $(1, -4)$
replace $x = 3 \implies f''(3) = 72 > 0$, relative minimum at $(0, -36)$

Note: These results are the same as in Example 1 and the graph of Example 1.

Example 6: Given $f(x) = \dfrac{x+2}{2x^2}$, find:

the relative extrema using the second derivative test.

$f'(x) = -\dfrac{x+4}{2x^3}$: numerator$= 0 \implies x + 4 = 0 \implies x = -4$; denominator$= 0 \implies 2x^3 = 0 \implies x = 0$

the critical numbers are $x = -4$ and $x = 0$

$f''(x) = \dfrac{x+6}{x^4}$

replace $x = -4 \implies f''(-4) = \frac{1}{128} > 0$, relative minimum at $\left(-4, -\frac{1}{16}\right)$

replace $x = 0 \implies f''(0) = \text{undefined}$, no extrema since the y value is undefined,
a vertical asymptote at $x = 0$

Note: These results are the same as in Example 2 and the graph of Example 2.

Absolute Extrema

Given $f(x)$ over a closed interval $[a,b]$, find the highest y value in that interval called the absolute maximum and the lowest y value in that interval called the absolute minimum.
First we locate any C.N. in that interval using the first derivative $f'(x)$, then we build a table of values including the boundaries of the interval and any C.N. in that interval. The table has the x values and the corresponding y values. The highest y will be the absolute maximum and the lowest y will be the absolute minimum.

Example 7: Given $f(x) = x^3 - 27x$ over $[0,5]$

$f'(x) = 3x^2 - 27 = 3(x-3)(x+3)$

the critical numbers are $x = -3$; $x = 3$; only $x = 3$ is between $x = 0$ and $x = 5$

x	$y = f(x)$
0	$f(0) = 0 \implies$ absolute maximum
3	$f(3) = -54 \implies$ absolute minimum
5	$f(5) = -10$

Example 8: Given $f(x) = \dfrac{x-2}{(x+1)^2}$ on $[-2,6]$

$f'(x) = \dfrac{5-x}{(x+1)^3}$

numerator$= 0 \implies 5 - x = 0 \implies x = 5$; denominator$= 0 \implies (x+1)^3 = 0 \implies x = -1$
the critical numbers are $x = -1$; $x = 0$; both values are between $x = -2$ and $x = 6$

x	$y = f(x)$
-2	$f(-2) = -4$
-1	$f(-1) = -\infty$
5	$f(5) = \frac{1}{12} \approx 0.0833$
6	$f(6) = \frac{4}{49} \approx 0.0816$

absolute maximum is $\frac{1}{12} \approx 0.0833$ at $x = 5$; no absolute minimum.

Curve Sketching

To sketch the curve of $f(x)$ using the derivative method implies following these steps:
(1) $f(x)$ gives $x-y$ intercepts, any asymptotes if $f(x)$ is a rational function.
(2) $f'(x)$ gives relative extrema, intervals of increasing & decreasing with the first derivative test.
(3) $f''(x)$ gives point(s) of inflection, intervals of concavity with the second derivative test.
(4) a table including all results from steps (1) to (3).
(5) Sketch the curve of $f(x)$ showing all points found in steps (1) to (3).

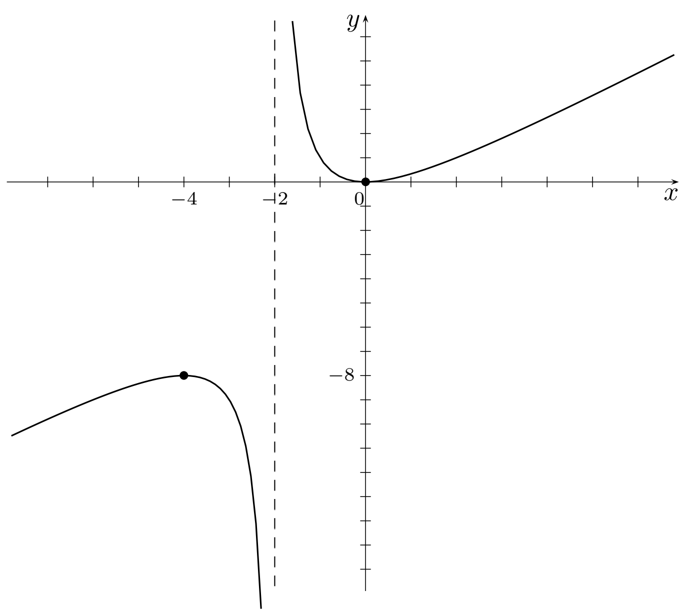
Example 9: Given $f(x) = \dfrac{x^2}{x+2}$; list (if any) all x and y intercepts, vertical and horizontal asymptotes, relative extrema, points of inflection, intervals where $f(x)$ is increasing, decreasing, concave up, concave down and then sketch the graph of $f(x)$.

from $f(x) \implies x-y$ intercept: $(0,0)$
asymptotes: vertical asymptote: denominator$=0 \implies x+2=0 \implies x=-2$
horizontal asymptote: $\lim\limits_{x\to\infty} \dfrac{x^2}{x+2}$ =D.N.E. ; no horizontal asymptote

from $f'(x) = \dfrac{x(x+4)}{(x+2)^2}$: the critical point at: $x=0$, $y=0$

from $f''(x) = \dfrac{8}{(x+2)^3}$: no point of inflection

$x-$values:	$-\infty$		-4		-2		0		$+\infty$
$f'(x)$		$+$		$-$		$-$		$+$	
$f''(x)$		$-$		$-$		$+$		$+$	
$y-$values:	$-\infty$		-8		und		0		∞
shape:		inc.down	Max	dec.down	VA	dec.up	Min	inc.up	



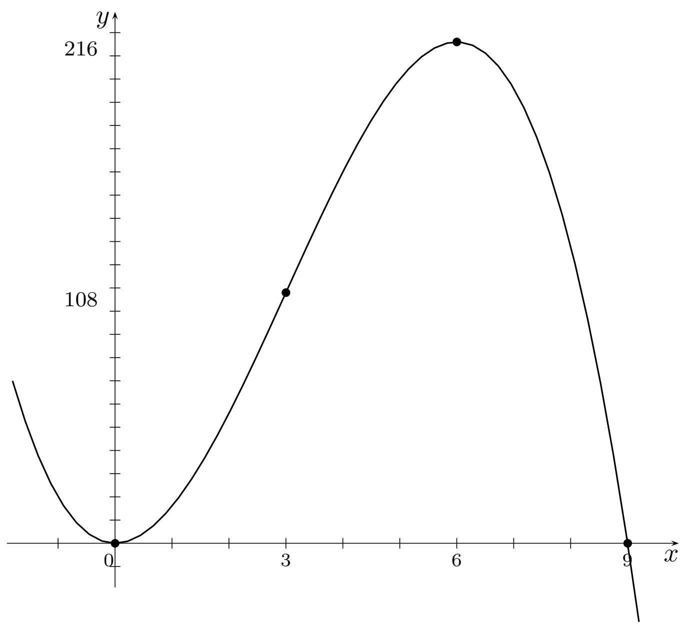
Example 10: Given $f(x) = 18x^2 - 2x^3$; list (if any) all x and y intercepts, vertical and horizontal asymptotes, relative extrema, points of inflection, intervals where $f(x)$ is increasing, decreasing, concave up, concave down and then sketch the graph of $f(x)$.

from $f(x) = 2x^2(9 - x) \implies x - y$ intercept: $(0, 0)$; $(9, 0)$
no asymptotes since $f(x)$ is a polynomial.

from $f'(x) = 36x - 6x^2 = 6x(6 - x)$: the critical points at: $(0, 0)$; $(6, 216)$

from $f''(x) = 36 - 12x$: possible point of inflection at $(3, 108)$

$x-$values:	$-\infty$		0		3		6		$+\infty$
$f'(x)$		$-$		$+$		$+$		$-$	
$f''(x)$		$+$		$+$		$-$		$-$	
$y-$values:	$+\infty$		0		108		216		$-\infty$
shape:		dec.up		inc.up		inc.down		dec.down	
			Min		P.I.		Max		



Example 11: Sketch $f(x)$ with the following conditions:

points at $(-1,1)$, $(0,0)$, $(1,-1)$; horizontal asymptote: $\lim_{x\to\infty} f(x) = 0$

$x < -1 \implies f'(x) < 0$, $f''(x) > 0$
$-1 < x < 0 \implies f'(x) < 0$, $f''(x) > 0$
$0 < x < 1 \implies f'(x) < 0$, $f''(x) < 0$
$x > 1 \implies f'(x) > 0$, $f''(x) < 0$

intervals x values	signs of $f'(x)$	signs of $f''(x)$	shape of $f(x)$ and y values
$-\infty$			
$]-\infty,-1[$	$-$	$+$	decreasing, concave up
-1			$(-1,1)$
$]-1,0[$	$-$	$-$	decreasing, concave down
0			point of inflection at $(0,0)$
$]0,1[$	$-$	$-$	decreasing, concave down
1			relative minimum at $(1,-1)$
$]1,+\infty[$	$+$	$-$	increasing, concave down
$+\infty$			horizontal asymptote at $y=0$