Exponential Function

The general form of the exponential function equation is

$$\boxed{y = f(x) = a^x}$$

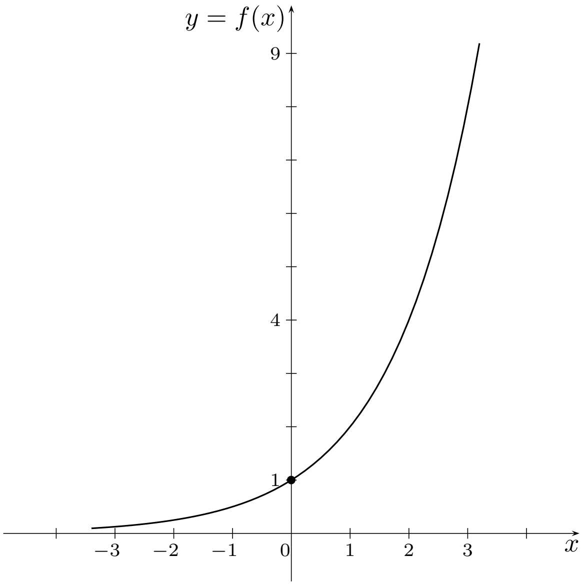
where a is a real number called the base of the exponential function and the exponent of the base is the variable x.

The base a cannot equal 1 and is also a positive number greater than 0.

Observe the next 3 graphs, the behavior of each function is that as x value increases, y value increases rapidly.

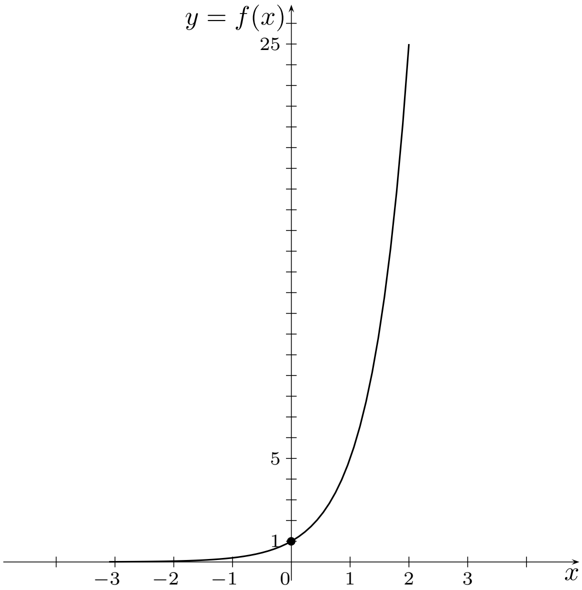
Graph 1 of the Exponential Function

x	$y = f(x) = 2^x$
-2	0.25
-1	0.5
0	1
1	2
2	4
3	9



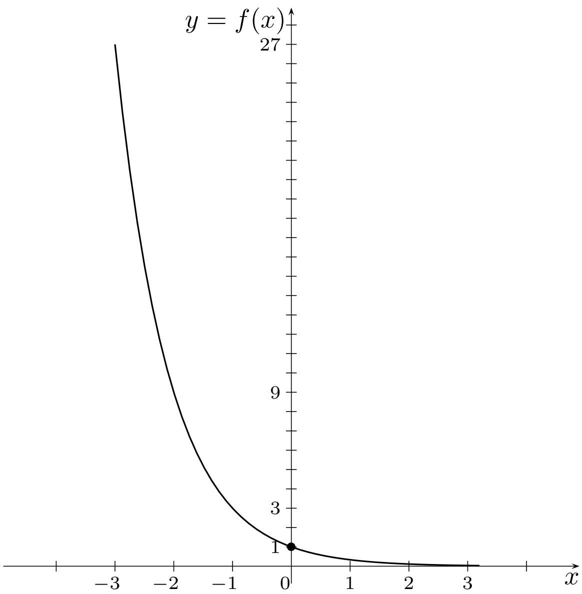
Graph 2 of the Exponential Function

x	$y = f(x) = 5^x$
-2	0.04
-1	0.2
0	1
1	5
2	25
3	125



Graph 3 of the Exponential Function

x	$y = f(x) = \left(\frac{1}{3}\right)^x = 3^{-x}$
-3	27
-2	9
-1	3
0	1
1	0.33
2	0.11



Properties of Exponential Function

(1) $a^0 = 1$

(2) $a^{-1} = \dfrac{1}{a}$

(3) $\dfrac{a^n}{a^m} = a^{n-m}$

(4) $a^m \,.\, a^n = a^{m+n}$

(5) $(a^m)^p = a^{m \cdot p}$

(6) $a^{+\infty} = +\infty$

(7) $a^{-\infty} = 0$

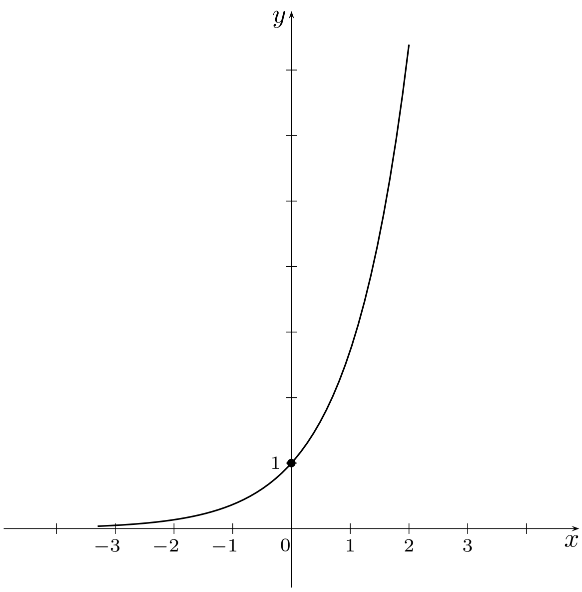
(8) $\dfrac{1}{a^n} = a^{-n}$

Common bases for Exponentials

The most common base is $''e''$ an irrational number like π and the decimal value is $e \approx 2.71828\ldots$

the calculator has 2 keys for 2 common bases: base 10 has key 10^x and base e has key e^x .

The graph of $y = e^x$ is shown below.



Examples for Exponentials

(a) Reduce: $e^4 \,.\, e^{-5} \,.\, e^{1/2} = e^{4-5+1/2} = e^{-1/2}$

(b) Simplify $\dfrac{4^{3x} \,.\, 4^{5x}}{4^{2x} \,.\, 4^{-x}} = 4^{3x+5x-(2x-x)} = 4^{7x}$

(c) Expand: $\left(e^{2x} + e^{-x}\right)^2 = \left(e^{2x}\right)^2 + 2\,e^{2x}\,e^{-x} + \left(e^{-x}\right)^2 = e^{4x} + 2\,e^{x} + e^{-2x}$

(d) Multiply: $(3^x + 3) \,.\, (9 + 3^x)$
$9 \,.\, 3^x + 3^x \,.\, 3^x + 9 \,.\, 3 + 3 \,.\, 3^x = 3^2 \,.\, 3^x + 3^{2x} + 27 + 3^{1+x} = 3^{2+x} + 3^{2x} + 27 + 3^{1+x}$

Inverse Functions

The inverse operation consists of interchanging variables;

therefore interchanging domain and range.

If we apply the inverse operation to the exponential function $y = a^x$,

we get $x = a^y$ and this equation is called the exponential notation of the logarithmic function.

The domain of the exponential function is all reals and the range of the exponential function is greater than 0.

Inversely, the domain of the logarithmic function is greater than 0 and the range of the logarithmic function is all reals.

Note: the behavior of the exponential function is that as x value increases, y value increases rapidly.

Inversely, as x value increases the y value increases slowly for the logarithmic function.

Logarithmic Function:

The general form of the logarithmic function equation has 2 shapes: exponential notation and the logarithmic notation as shown below:

$$\boxed{x = a^y \Longleftrightarrow y = \log_a(x)}$$

where a is a real number called the base of the logarithmic function.

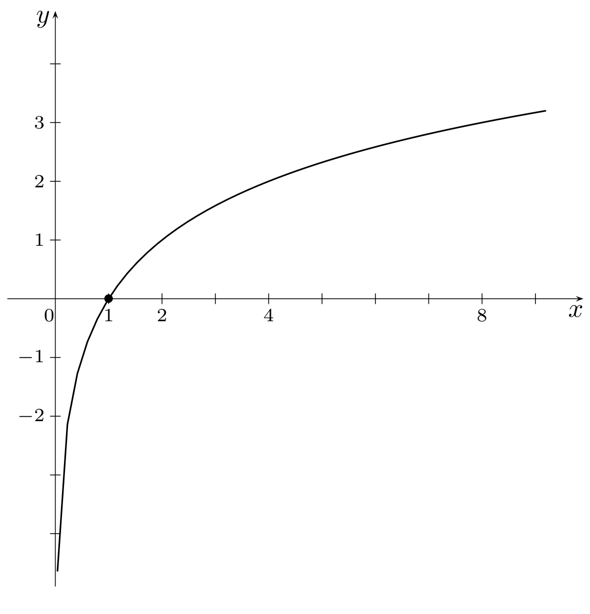
The base a cannot equal 1 and is also a positive number greater than 0.

Observe the next 3 graphs, the behavior of each function is that as x value increases, y value increases slowly.

Graph 1 of the Logarithmic Function

$x = 2^y \Longleftrightarrow y = \log_2(x)$

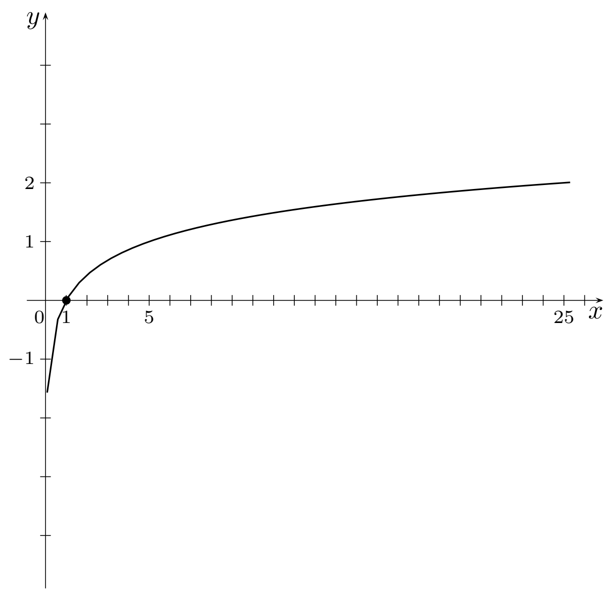
x	$y = f(x) = \log_2(x)$
0.25	-2
0.5	-1
1	0
2	1
4	2
8	3



Graph 2 of the Logarithmic Function

$x = 5^y \iff y = \log_5(x)$

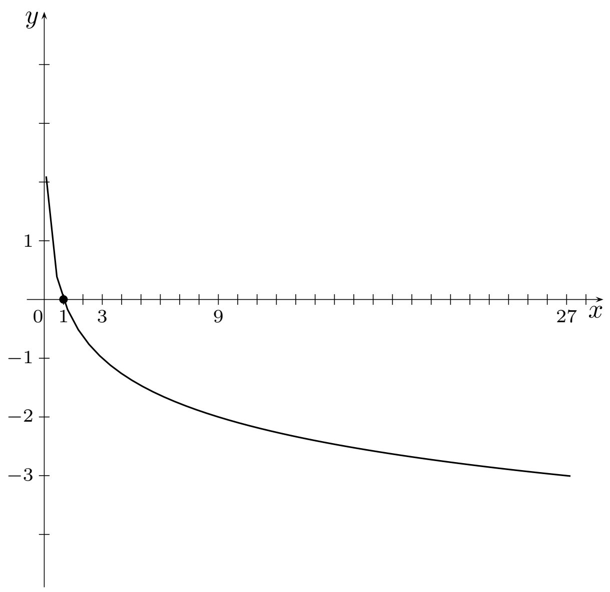
x	$y = f(x) = \log_5(x)$
0.04	-2
0.2	-1
1	0
5	1
25	2
125	3



Graph 3 of the Logarithmic Function

$x = 3^{-y} \iff y = -\log_3(x)$

x	$y = f(x) = -\log_3(x)$
27	-3
9	-2
3	-1
1	0
0.33	1
0.11	2



Properties of Logarithmic Function

(1) $\log_a (1) = 0$

(2) $\log_a (0^+) = -\infty$

(3) $\log_a \left(\dfrac{m}{n}\right) = \log_a (m) - \log_a (n)$

(4) $\log_a (m \, . \, n) = \log_a (m) + \log_a (n)$

(5) $\log_a (m^p) = p \, \log_a (m)$

(6) $\log_a (+\infty) = +\infty$

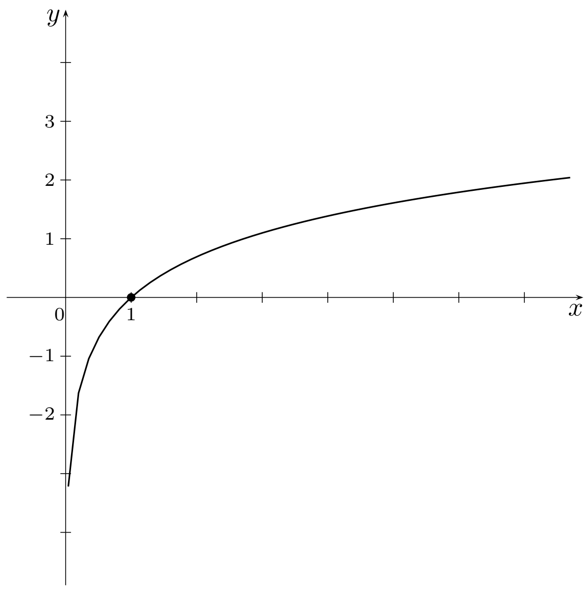
(7) $\log_a \left(m^{-1}\right) = \log_a \left(\dfrac{1}{m}\right) = -\log_a (m)$

Common bases for Logarithms

The most common base is $''e''$ an irrational number like π and the decimal value is $e \approx 2.71828\ldots$

the calculator has 2 keys for 2 common bases: base 10 has key $\log(x)$ and base e has key $\ln(x)$.

The graph of $y = \ln(x)$ is shown below.



Examples for Logarithms

(a) Reduce: $\log_3 (x+1) + \log_3 (x-1) = \log_3 (x^2-1)$

(b) Simplify $\log_5 (x) - \log_5 (x-1) = \log_5 \left(\dfrac{x}{x-1}\right)$

(c) Expand into simple logs: $\log_6 \left(\dfrac{x^3}{x+2}\right) = \log_6 \left(x^3\right) - \log_6 (x+2) = 3 \log_6 (x) - \log_6 (x+2)$

(d) Simplify: $2 \log_8 (x) - \log_8 (x^3+3x^2)$

$\log_8 \left(x^2\right) - \log_8 \left[x^2(x+3)\right] = \log_8 \left(x^2\right) - \log_8 \left(x^2\right) - \log_8 (x+3) = -\log_8 (x+3))$

Properties of Exponentials & Logarithms

(1) $\log_a (a^x) = x \Longleftrightarrow \log_e (e^x) = \ln(e^x) = x$

(2) $a^{\log_a (x)} = x \Longleftrightarrow e^{\log_e (x)} = e^{\ln(x)} = x$

Examples:

(1) $10^{\log_{10} (x^2)} = 10^{\log (x^2)} = x^2$

(2) $\log_e \left(e^{-4}\right) = \ln\left(e^{-4}\right) = -4$

(3) Simplify: $\log_4 \left(4^{x^2}\right) = x^2$

(4) Simplify: $5^{3 \log_5 (x)} = 5^{\log_5 \left(x^3\right)} = x^3$

(5) Simplify: $e^{-\ln\left(x^2\right)} = e^{\ln\left(x^{-2}\right)} = x^{-2} = \dfrac{1}{x^2}$

(6) Simplify: $2 \ln\left(e^{-x^2}\right) = \ln\left(e^{-x^2}\right)^2 = \ln\left(e^{-2x^2}\right) = -2x^2$

Graph of Exponential & Logarithmic Functions

The graph below shows the exponential function (brown curve) $y = 2^{-x}$ and its inverse equation

the logarithmic function (blue curve) $x = 2^{-y}$ which is the same as $y = -\log_2(x)$.

Notice the symmetry of both curves with the (red) line $y = x$.

Also point $(-1, 2)$ on the exponential function has its symmetrical point $(2, -1)$ on logarithmic function.

The domain of exponential function is all reals becomes the range of logarithmic function;

also the range of exponential function is greater than 0 becomes the domain of logarithmic function.