Topics of Functions used in Calculus

Definition of a function $f(x)$:

A function $y = f(x)$ is a relation between 2 variables $x$ and $y$ such that for every $x$ value only one $y$ value.

A point is represented by the coordinates $(x, y)$.

If we have a set of points like $(-1, 2)$ ; $(1, 3)$ ; $(3, 2)$ ; $(4, -1)$ ;
these points represent by definition a function where the domain of the function is the set of $x$ values $\{-1, 1, 3, 4\}$ and the range of the function is the set of the corresponding $y$ values $\{-1, 2, 3\}$.

Most of the time, a function is presented by an equation.
The graph of a function $y = f(x)$ may have intercepts with the axes.

When $x = 0$, we get the $y$ intercept and when $y = 0$, we get the $x$ intercept.

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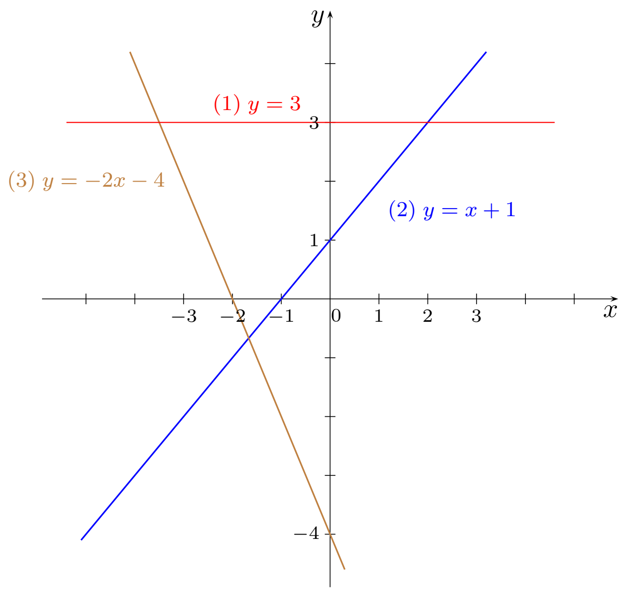
Linear function

A linear function has the general function $y = ax + b$

The graph shows 3 linear functions called lines:
(1) $y = 3$ represents a horizontal line (red)
(2) $y = x + 1$ (blue line)
(3) $y = -2x - 4$ (brown line)




Domain of the 3 lines is $\Re$ (all real numbers)
Range of (1) is $\{3\}$.
Range of (2) is $\Re$.
Range of (3) is $\Re$.
Even if the window of the graph restricts the view of the domain and range, the lines still go on.

Line (1) shows a horizontal line
$y = 0x + 3 = 3$ with slope $m = 0$
and $y$ intercept $(0, 3)$, no $x$ intercept.

Line (2) shows $y = x + 1$ with slope $m = 1$,
$x-$intercept $(-1, 0)$ and $y-$intercept $(0, 1)$

Line (3) shows $y = -2x - 4$ with slope $m = -2$,
$x-$intercept $(-2, 0)$ and $y-$intercept $(0, -4)$

Equation of a Line:

The general form of an equation of a line is $y = mx + b$

To find the equation, need one of 2 next situations:

(1) coordinates of point $(x_1\,,\,y_1)$ on the line and slope $m$

(2) coordinates of 2 points $(x_1\,,\,y_1)$ and $(x_2\,,\,y_2)$

Example 1:

Given point $(2, 3)$ and slope 4, find the equation of the line.

The form used is: $y - y_1 = m\,(x - x_1)$

replace and we get: $y - 3 = 4\,(x - 2) \implies y = 4x - 5$

The slope is $m = 4$ and $b = -5$ ($y-$intercept)

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Example 2:

Given 2 points $(3, 1)$ $(-1, 2)$ , find the equation of the line.

To find the slope $m = \dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{1 - 2}{3 - (-1)} = -\dfrac{1}{4}$

The form used is: $y - y_1 = m\,(x - x_1)$

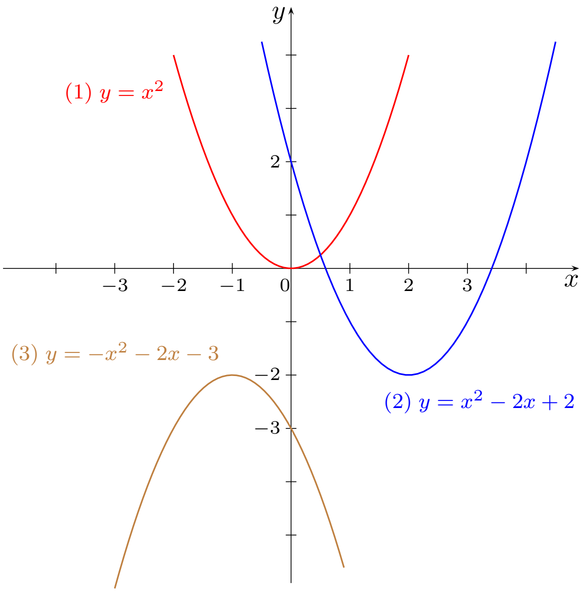
replace and we get: $y - 2 = -\frac{1}{4}\,(x + 1) \implies y = -\frac{1}{4}\,x + \frac{7}{4} \implies 4y + x = 7$

The slope is $m = -\dfrac{1}{4}$ and $b = \dfrac{7}{4}$ ($y-$intercept)

Quadratic function

A quadratic function has the general function $y = ax^2 + bx + c$

The graph shows 3 quadratic functions
called parabolas:
(1) $y = x^2$ (red parabola)
(2) $y = x^2 - 4x + 2$ (blue parabola)
(3) $y = -x^2 - 2x - 3$ (brown parabola)




Domain of the 3 parabolas is $\Re$ (all real numbers)
Range of (1) is $y \geq 0$ .
Range of (2) is $y \geq -2$ .
Range of (3) is $y \leq -2$ .
Even if the window of the graph restricts the view of the domain and range, the parabolas still go on.

Parabola (1) shows $y = x^2$ with
vertex at $(0, 0)$ opening upwards
$x-$intercept $(0, 0)$ and $y-$intercept $(0, 0)$ .

Parabola (2) shows $y = x^2 - 4x + 2$ with
vertex at $(2, -2)$ opening upwards
$x-$intercepts $(0.6, 0)$ and $(3.4, 0)$ ; $y-$intercept $(0, 2)$ .

Parabola (3) shows $y = -x^2 - 2x - 3$ with
vertex at $(-1, -2)$ opening downwards
no $x-$intercept and $y-$intercept $(0, -3)$

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Example:

Given the curve of $y = 12x - 3x^2$ , find the vertex and the intercepts.

Use the quadratic function $y = ax^2 + bx + c$ with a vertex at $x = -\dfrac{b}{2a}$ and find the corresponding $y$ value;

$x-$intercept: make $y = 0$ and find the $x$ value(s)

$y-$intercept: make $x = 0$ and find the $y$ value

In this example, vertex is located at $x = -\dfrac{12}{2(-3)} = 2$ , $y = 12(2) - 3(2)^2 = 12$

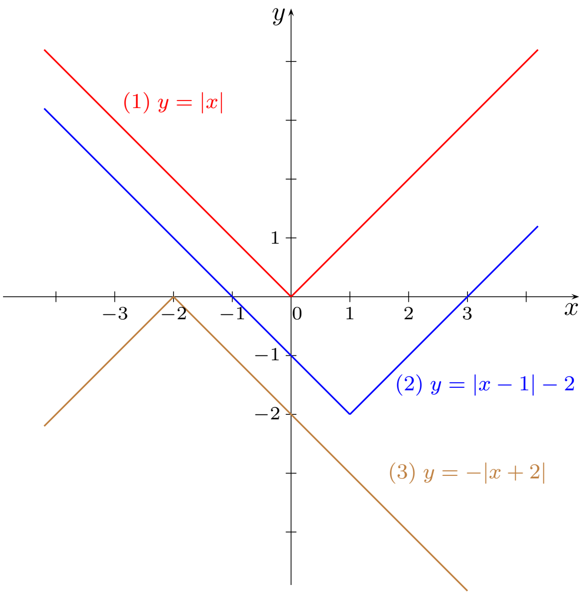
so the vertex is at (2,12) , opening downwards since $a = -3 < 0$ ,

$x-$intercepts at $(0, 0)$ and $(4, 0)$ ; $y-$intercept at $(0, 0)$

Absolute Value function

A absolute value function has the general function $y = a\,|x-h| + k$

The graph shows 3 absolute value functions called $V-$graphs:
(1) $y = |x|$ (red $V-$graph)
(2) $y = |x-1| - 2$ (blue $V-$graph)
(3) $y = -|x+2|$ (brown $V-$graph)




Domain of the 3 $V-$graphs is $\Re$ (all real numbers)
Range of (1) is $y \geq 0$.
Range of (2) is $y \geq -2$.
Range of (3) is $y \leq 0$.
Even if the window of the graph restricts the view of the domain and range, the $V-$graphs still go on.

$V-$graph (1) shows $y = |x|$ with
vertex at $(0,0)$ opening upwards
$x-$intercept $(0,0)$ and $y-$intercept $(0,0)$.

$V-$graph (2) shows $y = |x-1| - 2$ with
vertex at $(1,-2)$ opening upwards
$x-$intercepts $(-1,0)$ and $(3,0)$; $y-$intercept $(0,-1)$.

$V-$graph (3) shows $y = -|x+2|$ with
vertex at $(-2,0)$ opening downwards
$x-$intercept $(-2,0)$ and $y-$intercept $(0,-2)$

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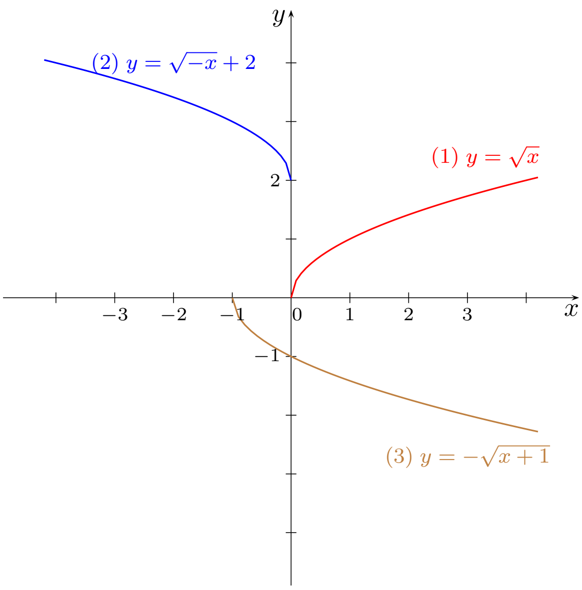
Square Root function

A square root function has the general function $y = a\sqrt{x-h} + k$

The graph shows 3 square root functions called irrational functions:
(1) $y = \sqrt{x}$ (red curve)
(2) $y = \sqrt{-x} + 2$ (blue curve)
(3) $y = -\sqrt{x+1}$ (brown curve)




Domain of (1) is $x \geq 0$; Range is $y \geq 0$.
Domain of (2) is $x \leq -2$; Range is $y \geq 2$.
Domain of (3) is $x \geq -1$; Range is $y \leq 0$.
Even if the window of the graph restricts the view of the domain and range, the irrational curves still go on.

curve (1) shows $y = \sqrt{x}$ has
$x-$intercept $(0,0)$ and $y-$intercept $(0,0)$.

curve (2) shows $y = \sqrt{-x} + 2$ has
no $x-$intercept and $y-$intercept $(0,2)$.

curve (3) shows $y = -\sqrt{x+1}$ has
$x-$intercept $(-1,0)$ and $y-$intercept $(0,-1)$

Rational function

A rational function has the general function $y = \dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials

(1) The graph shows a rational function called fractional expression:

$y = \dfrac{1}{x} = x^{-1}$

Domain: $\Re - \{0\} \longrightarrow x \neq 0$

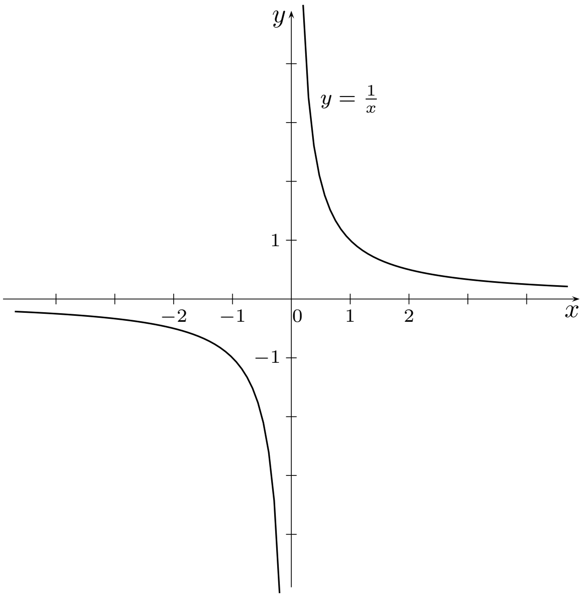
Range: $\Re - \{0\} \longrightarrow y \neq 0$

vertical asymptote: denominator$= 0 \implies x = 0$

horizontal asymptote: when $x \to \pm\infty \implies y = 0$

no $x$ and $y$ intercepts.

Even if the window of the graph restricts the view of the domain and range, the rational curve still go on.

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(2) The graph shows a rational function called fractional expression:

$y = \dfrac{2x-1}{x-1}$

Domain: $\Re - \{1\} \longrightarrow x \neq 1$

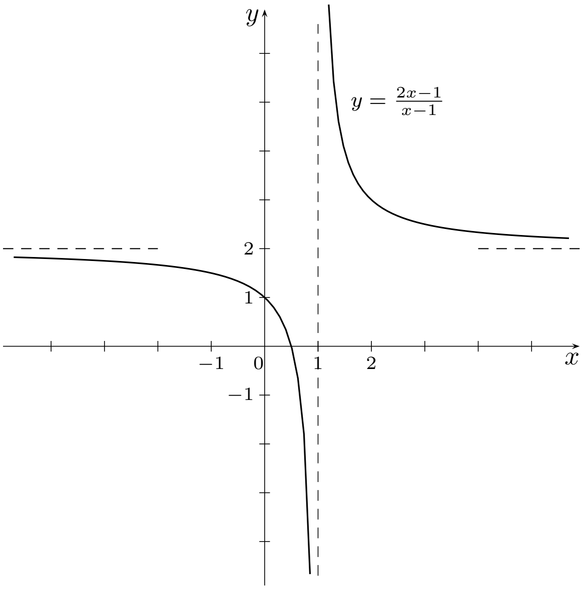
Range: $\Re - \{2\} \longrightarrow y \neq 2$

vertical asymptote: denominator$= 0 \implies x = 1$

horizontal asymptote: when $x \to \pm\infty \implies y = 2$

$x-$intercept: $\left(\frac{1}{2}, 0\right)$ and $y-$intercept: $(0, 1)$

Even if the window of the graph restricts the view of the domain and range, the rational curve still go on.



(3) The graph shows a rational function called fractional expression:

$$y = \frac{-2}{x+1}$$

Domain: $\Re - \{-1\} \longrightarrow x \neq -1$

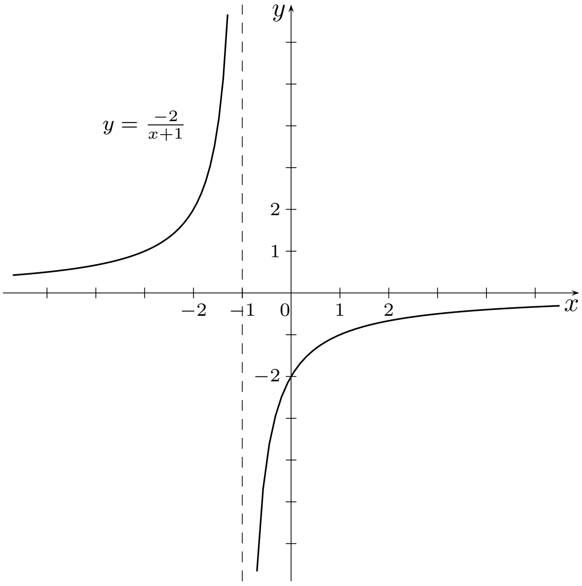
Range: $\Re - \{0\} \longrightarrow y \neq 0$

vertical asymptote: denominator$= 0 \implies x = -1$

horizontal asymptote: when $x \to \pm\infty \implies y = 0$

no $x-$intercept: and $y-$intercept: $(0, -2)$

Even if the window of the graph restricts the view of the domain and range, the rational curve still go on.

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<u>Piecewise function</u>

(1) The graph shows a piecewise function

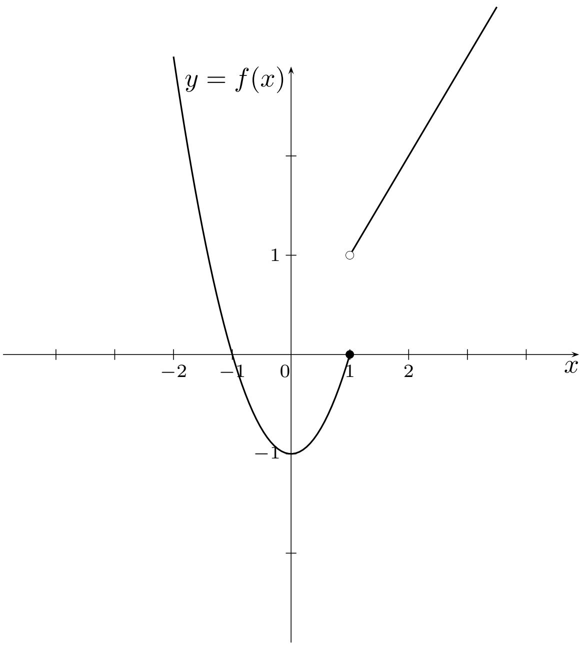
$$f(x) = \begin{cases} x^2 - 1 & \text{if} \quad x \leq 1 \\ x & \text{if} \quad x > 1 \end{cases}$$

Domain: $\Re$ ; Range: $y \geq 1$

The graph on the left of $x = 1$ is a piece of a parabola

The graph on the right of $x = 1$ is a piece of a line

Even if the window of the graph restricts the view of the domain and range, the piecewise curve still go on.

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(2) The graph shows a piecewise function

$$g(x) = \begin{cases} -x^2 + 4 & \text{if} \quad x \leq 1 \\ x - 1 & \text{if} \quad x > 1 \end{cases}$$

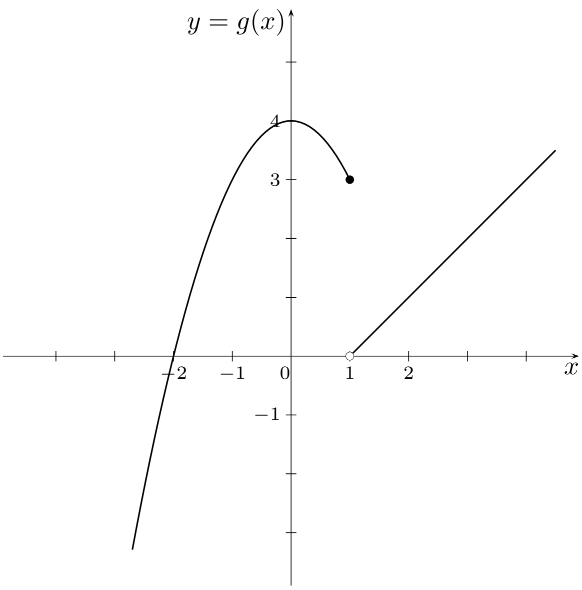
Domain: $\Re$ ; Range: $\Re$

$x-$intercept $(-2, 0)$ ; $y-$intercept $(0, 4)$

The graph on the left of $x = 1$ is a piece of a parabola

The graph on the right of $x = 1$ is a piece of a line

Even if the window of the graph restricts the view of the domain and range, the piecewise curve still go on.



(3) The graph shows a piecewise function

$$h(x) = \begin{cases} -x-1 & \text{if} \quad x \le -1 \\ -x^2+1 & \text{if} \quad -1 < x \le 2 \\ x & \text{if} \quad x > 2 \end{cases}$$

Domain: $\Re$ ; Range: $y \ge -3$

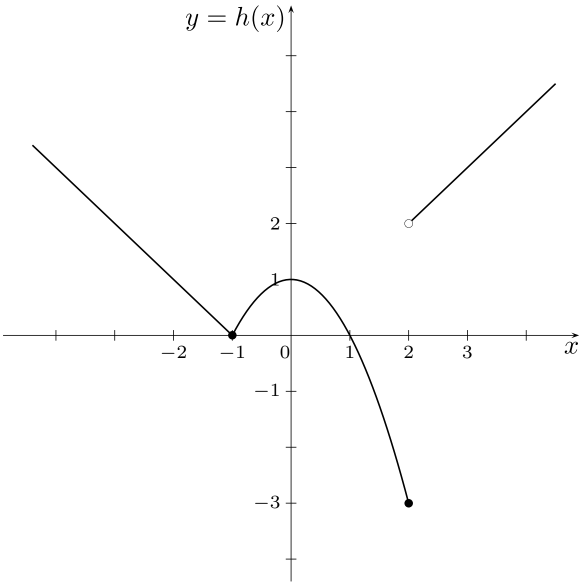
$x-$intercepts $(-1, 0)$ and $(1, 0)$ ; $y-$intercept $(0, 1)$

The graph on the left of $x = -1$ is a piece of a line

The graph between $x = -1$ and $x = 2$ is a piece of a parabola

The graph on the right of $x = 2$ is a piece of a line

Even if the window of the graph restricts the view of the domain and range, the piecewise curve still go on.

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Operations on functions

There are 6 operations on functions that can be done:

(1) addition ; (2) subtraction ; (3) multiplication ; (4) division ; (5) composition ; (6) inverse

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(make sure that every answer is reduced)

(1) add $f(x) = x^2 + 1$ to $g(x) = 2x - 3$
$(f + g)(x) = f(x) + g(x) = (x^2 + 1) + (2x - 3) = x^2 + 2x - 2$

(2) subtract $f(x) = 4x - 5$ from $g(x) = x^2 - x + 1$
$(f - g)(x) = f(x) - g(x) = (x^2 - x + 1) - (4x - 5) = x^2 - 5x + 6$

(3) multiply $f(x) = 3x - 1$ with $g(x) = -x^2 + x$
$(f \, . \, g)(x) = f(x) \, . \, g(x) = (3x - 1) \, . \, (-x^2 + x) = -3x^3 + x^2 + 3x^2 - x = -3x^3 + 4x^2 - x = -3x^3 + 4x^2 - x$

(4) divide $f(x) = 3x$ by $g(x) = x + 1$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{3x}{x+1}$$

(5) composition $f\Big(g(x)\Big) = (f \circ g)(x)$ or $g\Big(f(x)\Big) = (g \circ f)(x)$

Given $f(x) = 2x^2 - 4x$ and $g(x) = x + 5$, find $(f \circ g)(x)$.

replace every $x$ in $f(x)$ with $g(x)$ as follows:

$$(f \circ g)(x) = f\Big(g(x)\Big) = 2(x+5)^2 - 4(x+5) = 2(x^2 + 10x + 25) - 4x - 20 = 2x^2 + 16x + 30$$

<u>Operations on functions</u>

(5) composition $f\Big(g(x)\Big) = (f \circ g)(x)$ or $g\Big(f(x)\Big) = (g \circ f)(x)$

Given $f(x) = x^2 - 2x$ and $g(x) = 3x + 5$, find $(g \circ f)(x)$.

replace every $x$ in $g(x)$ with $f(x)$ as follows:

$$(g \circ f)(x) = g\Big(f(x)\Big) = 3(x^2 - 2x) + 5 = 3x^2 - 6x + 5$$

(6) the inverse of $f(x)$ is denoted by $f^{-1}(x)$

To find the inverse of $f(x)$, interchange $x$ and $y$ and we get $x = f(y)$ then isolate $y$ to get $f^{-1}(x)$.

Given $f(x) = 2x - 1$, find $f^{-1}(x)$.

Let $y = 2x - 1$, interchange $x$ and $y$, we get:

$x = 2y - 1 \implies x + 1 = 2y \implies y = \dfrac{x+1}{2} = \frac{1}{2}\,x + \frac{1}{2}$, therefore the inverse of $f(x)$ is $f^{-1}(x) = \frac{1}{2}\,x + \frac{1}{2}$.

Note: It is possible that a function has an inverse equation but no inverse function like $y = f(x) = x^2$ has the inverse equation $x = y^2$ but this equation is not a function.

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<u>Example 1:</u> Given $f(x) = x^2 - 3x + 7$, find:

(a) $f(2) \longrightarrow$ replace every $x$ in $f(x)$ by 2, we get: $f(2) = (2)^2 - 3(2) + 7 = 5$

(b) $f(x+h) - f(x)$
To get $f(x+h)$, replace every $x$ in $f(x)$ with $x + h$, then subtract $f(x)$.

$$\begin{aligned} f(x+h) - f(x) &= \big[(x+h)^2 - 3(x+h) + 7\big] - \big[x^2 - 3x + 7\big] \\ &= x^2 + 2xh + h^2 - 3x - 3h + 7 - x^2 + 3x - 7 = 2xh + h^2 - 3h \end{aligned}$$

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<u>Example 2:</u> Given $f(x) = x^3 + 2x - 1$, find: $f(1) - f(0)$

$f(1) = (1)^3 + 2(1) - 1 = 2$ and $f(0) = (0)^3 + 2(0) - 1 = -1 \implies f(1) - f(0) = 2 - (-1) = 3$

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<u>Example 3:</u> Given $f(x) = \dfrac{-1}{x}$, find: $f(x+h) - f(x)$

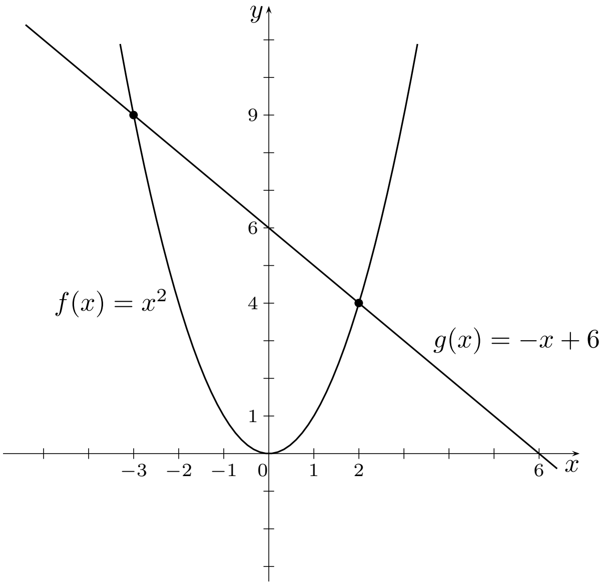
To get $f(x+h)$, replace every $x$ in $f(x)$ with $x + h$, then subtract $f(x)$.

$$f(x+h) - f(x) = \frac{-1}{x+h} - \frac{-1}{x} = \frac{-x}{x(x+h)} - \frac{-(x+h)}{x(x+h)} = \frac{-x + (x+h)}{x(x+h)} = \frac{h}{x(x+h)}$$

Intersection of 2 functions

Example 1:

Given $f(x) = x^2$ and $g(x) = -x + 6$ , find the points of intersection




Let $f(x) = g(x) \implies x^2 = -x + 6$
$x^2 + 6x - 6 = 0$
This quadratic can be solved by factoring
or by the quadratic formula.
The solution by factoring is $(x + 3)(x - 2) = 0$
$x = -3$ and $x = 2$
The corresponding $y$ values are
$x = -3 \rightarrow y = 9$ and $x = 2 \rightarrow y = 4$

Therefore the points of intersection are
$(-3, 9)$ and $(2, 4)$

The graph of both functions shows these points of intersection.

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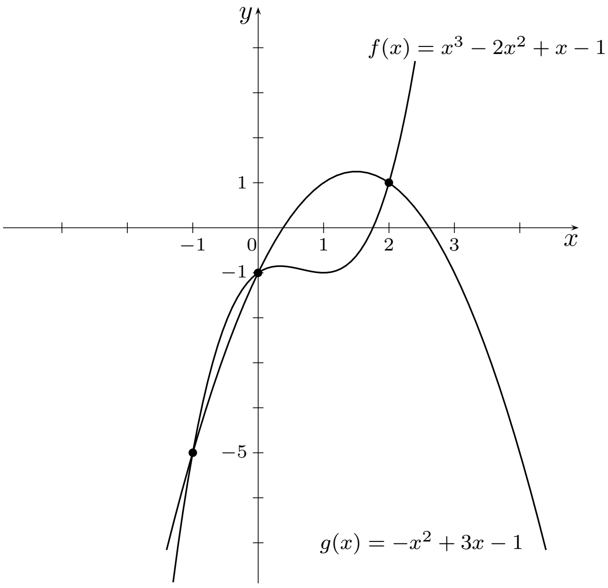
Example 2:

Given $f(x) = x^3 - 2x^2 + x - 1$ and $g(x) = -x^2 + 3x - 1$ , find the points of intersection




Let $f(x) = g(x)$
$x^3 - 2x^2 + x - 1 = -x^2 + 3x - 1$
$x^3 - x^2 - 2x = 0$
This equation can be solved by factoring
$x(x + 1)(x - 2) = 0$
$x = -1$ ; $x = 0$ and $x = 2$
The corresponding $y$ values are
$x = -1 \rightarrow y = -5$ ; $x = 0 \rightarrow y = -1$
and $x = 2 \rightarrow y = 1$

Therefore the points of intersection are
$(-1, -5)$ ; $(0, -1)$ and $(2, 1)$

The graph of both functions shows these points of intersection.

Division of 2 Polynomial functions

Given $\dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomial functions.

To divide, the degree of $P(x)$ must be greater than or equal to the degree of $Q(x)$.

If the degree of $P(x)$ is less than the degree of $Q(x)$ ; cannot divide.

Once the long division is done, we write the result as:

$$\boxed{\frac{P(x)}{Q(x)} = \text{result} + \frac{\text{remainder}}{Q(x)}}$$

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Example 1: $\dfrac{3x^2+3x-5}{x+1}$, the degree of the numerator is 2 and the degree of the denominator is 1.

We must use the long division as shown:

$$\begin{array}{r|l} & \quad 3x \\ \hline x+1 \;) & 3x^2+3x-5 \\ & -(\,3x^2+3x\;) \\ \hline & \qquad\qquad -5 \end{array}$$

Thus $\dfrac{3x^2+3x-5}{x+1} = 3x + \dfrac{-5}{x+1}$

where result$= 3x$ and remainder$= -5$

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Example 2: $\dfrac{2x-7}{x^2+4}$,

the degree of the numerator is 1 and the degree of the denominator is 2. We cannot divide.