Implicit Differentiation - Logarithmic Differentiation

Implicit versus Explicit:

An explicit equation is the kind like

(a) $y = 3x^2 + 5$; (b) $y = \sqrt{1 + x^3}$; (c) $y = \dfrac{2x}{4 - x^2}$; (d) $y = \ln(3x + 7)$

Notice that the explicit equation has on Left Hand Side y and on Right Hand Side an expression with x only. It is clear that the variable y is a function of variable x.

An implicit equation is the kind like

(e) $y^2 + xy = \sqrt{y}$; (f) $\ln(x^2 + y^2) = 2x + 3y$; (g) $e^{xy} + \dfrac{x}{y} = 4y$; (h) $2x^2 - 4y^3 = 5xy$

Notice that the implicit equation have both variables x and y can be mixed on both sides of the equation. Also the variable y is inside another function like in (f).

Derivative for Implicit equation

Example 1:

Given $3x - y^2 = x^2 + y$, find $\dfrac{dy}{dx}$ or y'

The technique is to find the derivative of each term in the equation. Keep in mind that y is a function of x; so the derivative of x is 1 but the derivative of y is y'. In the table below, each term has its derivative:

$3x \implies 3$
$-y^2 \implies -2yy'$
$x^2 \implies 2x$
$y \implies y'$

The derivative equation:

$3 - 2yy' = 2x + y'$

$-2yy' - y' = 2x - 3$

$-y'(2y + 1) = 2x - 3$

$$\frac{dy}{dx} = y' = \frac{2x - 3}{-(2y + 1)} = \frac{3 - 2x}{2y + 1}$$

Notice: The answer of the derivative is in terms of x and y. Most implicit equations are not functions.

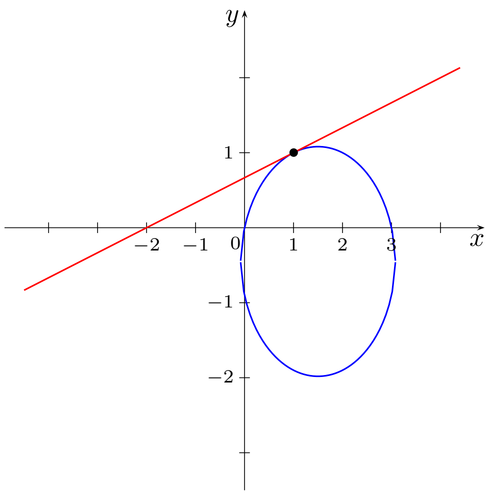
Find the equation of the tangent line at point $(1, 1)$

slope: $m = \left.\dfrac{dy}{dx}\right|_{(1,1)} = y'(1, 1)$

$m = \dfrac{3 - 2(1)}{2(1) + 1} = \dfrac{1}{3}$

Equation of the tangent line at point $(1, 1)$:

$y - 1 = \frac{1}{3}(x - 1) \implies y = \frac{1}{3}x + \frac{2}{3}$



Example 2:

Given $xy - 4 = y^2 - 6x$, find the equation of the tangent line at point $(1, -1)$

$xy \implies (1)y + (y')x$
$-4 \implies 0$
$y^2 \implies 2yy'$
$-6x \implies -6$

The derivative equation:

$y + xy' = 2yy' - 6$

$xy' - 2yy' = -y - 6$

$y'(x - 2y) = -y - 6$

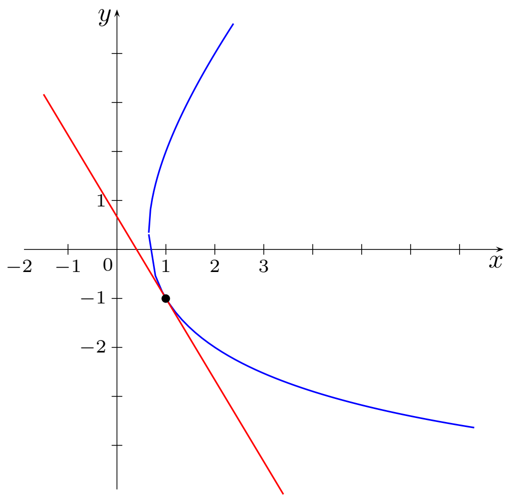
$$\frac{dy}{dx} = y' = \frac{-y - 6}{x - 2y}$$

slope: $m = \left.\frac{dy}{dx}\right|_{(1,-1)} = y'(1, -1)$

$$m = \frac{-(-1) - 6}{(1) - 2(-1)} = \frac{-5}{3}$$

Equation of the tangent line at point $(1, -1)$:

$$y - (-1) = -\frac{5}{3}(x - 1) \implies y = -\frac{5}{3}x + \frac{2}{3}$$



Example 3:

Given $\ln(x+y)+e^{y}=1$, find the equation of the tangent line at point $(1,0)$

$\ln(x+y) \implies \dfrac{1+y'}{x+y}$
$e^{y} \implies e^{y}\,y'$
$1 \implies 0$

The derivative equation:

$$\frac{1+y'}{x+y}+e^{y}\,y'=0$$

$$1+y'+(x+y)e^{y}\,y'=0$$

$$y'\,(1+x\,e^{y}+y\,e^{y})=-1$$

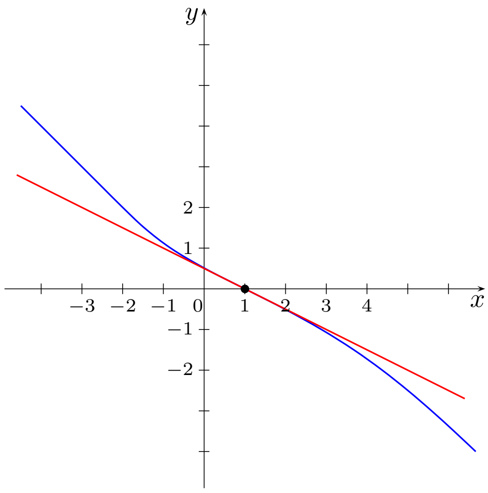
$$\frac{dy}{dx}=y'=\frac{-1}{1+x\,e^{y}+y\,e^{y}}$$

slope: $m=\left.\dfrac{dy}{dx}\right|_{(1,0)}=y'(1,0)$

$$m=\frac{-1}{1+(1)\,e^{0}+(0)\,e^{0}}=-\frac{1}{2}$$

Equation of the tangent line at point $(1,0)$:

$$y-0=-\tfrac{1}{2}\,(x-1) \implies y=-\tfrac{1}{2}\,x+\tfrac{1}{2}$$



Example 4:

Given $\sqrt{x} - y^2 = x - 3y$, find the equation of the tangent line at point $(4, 1)$

$\sqrt{x} \implies \dfrac{1}{2\sqrt{x}}$
$-y^2 \implies -2yy'$
$x \implies 1$
$-3y \implies -3y'$

The derivative equation:

$$\frac{1}{2\sqrt{x}} - 2yy' = 1 - 3y'$$

$$1 - 4\sqrt{x}\,yy' = 2\sqrt{x} - 6\sqrt{x}\,y'$$

$$y'\left(6\sqrt{x} - 4\sqrt{x}\,y\right) = 2\sqrt{x} - 1$$

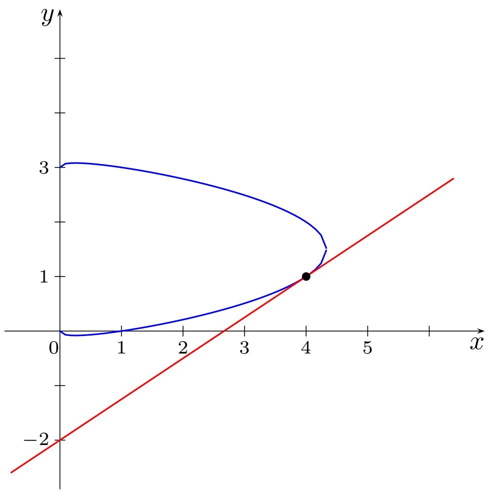
$$\frac{dy}{dx} = y' = \frac{2\sqrt{x} - 1}{6\sqrt{x} - 4\sqrt{x}\,y}$$

slope: $m = \left.\dfrac{dy}{dx}\right|_{(4,1)} = y'(4, 1)$

$$m = \frac{2\sqrt{4} - 1}{6\sqrt{4} - 4\sqrt{4}\,(1)} = \frac{3}{4}$$

Equation of the tangent line
at point $(4, 1)$:

$y - 1 = \frac{3}{4}(x - 4) \implies y = \frac{3}{4}x - 2$



Example 5:

Given $e^x + y = e^{xy} + 2$, find the equation of the tangent line at point $(0, 2)$

$e^x \implies e^x$
$y \implies y'$
$e^{xy} \implies e^{xy}\,(y + xy')$
$2 \implies 0$

The derivative equation:

$e^x + y' = e^{xy}\,(y + xy')$

$e^x + y' = y\,e^{xy} + x\,e^{xy}\,y'$

$y'\,(1 - x\,e^{xy}) = y\,e^{xy} - e^x$

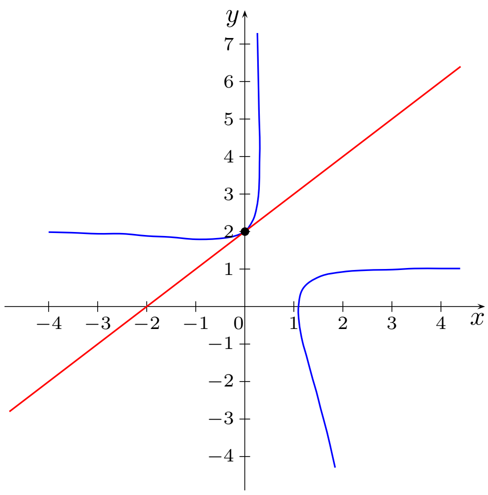
$$\frac{dy}{dx} = y' = \frac{y\,e^{xy} - e^x}{1 - x\,e^{xy}}$$

slope: $m = \left.\dfrac{dy}{dx}\right|_{(0,2)} = y'(0, 2)$

$$m = \frac{(2)\,e^{(0)(2)} - e^0}{1 - (0)\,e^{(0)(2)}} = 1$$

Equation of the tangent line
at point $(0, 2)$:

$y - 2 = (1)(x - 0) \implies y = x + 2$



Logarithmic Differentiation

This technique is used on two situations:
(1) The question says to use logarithmic differentiation.
(2) No other formulas, rules or techniques can be used.

To apply this method, take ln on both sides of the equation; then use implicit differentiation to get the derivative; finally isolate y' and give the answer in terms of x.

Example 6:

Use logarithmic differentiation to find the derivative of $y = \dfrac{x^3\,(x^2+1)^4}{\sqrt[3]{3x+2}}$

Take ln on both sides of the equation and use log properties:

$$\ln(y) = \ln\left[\frac{x^3\,(x^2+1)^4}{\sqrt[3]{3x+2}}\right] = \ln(x^3) + \ln(x^2+1)^4 - \ln(3x+2)^{1/3}$$

$$\ln(y) = 3\,\ln(x) + 4\,\ln(x^2+1) - \tfrac{1}{3}\,\ln(3x+2)$$

Use implicit differentiation and log formula derivatives:

$$\frac{1}{y}\,y' = 3\,\frac{1}{x} + 4\,\frac{2x}{x^2+1} - \tfrac{1}{3}\,\frac{3}{3x+2} \implies y' = y\left[\frac{3}{x} + \frac{8x}{x^2+1} - \frac{1}{3x+2}\right] = \frac{x^3\,(x^2+1)^4}{\sqrt[3]{3x+2}}\left[\frac{3}{x} + \frac{8x}{x^2+1} - \frac{1}{3x+2}\right]$$

The logarithmic differentiation is useful in this equation because without it you must use quotient rule,
then for the numerator you use the product rule and finally chain rule;
put all this together to get the derivative. It is long work and a messy expression that looks like this:

$$y' = \frac{\left[3x^2(x^2+1)^4 + 4(x^2+1)^3(2x)x^3\right]\,(3x+2)^{1/3} - \tfrac{1}{3}\,(3x-2)^{-2/3}(3)\,x^3\,(x^2+1)^4}{(3x+2)^{2/3}}$$

Example 7: Find the derivative of $y = x^{5x}$

We cannot use any previous formulas or rules.
x^n and a^x do not apply for this equation.
x^n means variable or function x to the power of a number n.
a^x means number a to the power of a variable or function x.
x^{5x} means variable or function of x to the power of a variable or function $5x$.
Therefore we have to rely to the logarithmic differentiation:
Take ln on both sides of the equation

$$\ln(y) = \ln\left(x^{5x}\right) \implies \ln(y) = 5x\,\ln(x)$$

The logarithmic differentiation is to use implicit differentiation on Left Hand Side and product rule on Right Hand Side as shown:

$$\frac{1}{y}\,y' = 5\,\ln(x) + 5x\,\frac{1}{x} = 5\,\ln(x) + 5 \implies y' = y\,[5\,\ln(x) + 5] = x^{5x}\,[5\,\ln(x) + 5]$$

No other technique gives this result except the logarithmic differentiation.
Very useful for a special situation.

Example 8:

Use logarithmic differentiation to find the derivative of $y = \sqrt{\dfrac{4x+1}{x^4\,(x^3+1)^2}}$

Take ln on both sides of the equation and use log properties:

$$\ln(y) = \ln\left[\sqrt{\frac{4x+1}{x^4\,(x^3+1)^2}}\right] = \tfrac{1}{2}\left[\ln(4x+1) - \ln(x^4) - \ln(x^3+1)^2\right]$$

$$\ln(y) = \tfrac{1}{2}\ \ln(4x+1) - 2\ \ln(x) - \ln(x^3+1)$$

Use implicit differentiation and log formula derivatives:

$$\frac{1}{y}\,y' = \tfrac{1}{2}\,\frac{4}{4x+1} - 2\,\frac{1}{x} - \frac{3x^2}{x^3+1} \implies y' = y\left[\frac{2}{4x+1} - \frac{2}{x} - \frac{3x^2}{x^3+1}\right] = \sqrt{\frac{4x+1}{x^4\,(x^3+1)^2}}\left[\frac{2}{4x+1} - \frac{2}{x} - \frac{3x^2}{x^3+1}\right]$$

Example 9: Find the derivative of $y = (x-1)^x$

We cannot use any previous formulas or rules.
x^n and a^x do not apply for this equation.
x^n means variable or function x to the power of a number n.
a^x means number a to the power of a variable or function x.
x^{5x} means variable or function of x to the power of a variable or function $5x$.
Therefore we have to rely to the logarithmic differentiation:
Take ln on both sides of the equation

$$\ln(y) = \ln\Big((x-1)^x\Big) \implies \ln(y) = x\ \ln(x-1)$$

The logarithmic differentiation is to use implicit differentiation on Left Hand Side and product rule on Right Hand Side as shown:

$$\frac{1}{y}\,y' = (1)\ \ln(x-1) + x\,\frac{x-1}{=}\,\ln(x-1) + \frac{x}{x-1} \implies y' = y\left[\ln(x-1) + \frac{x}{x-1}\right] = (x-1)^x\left[\ln(x-1) + \frac{x}{x-1}\right]$$

Example 10:

Use logarithmic differentiation to find the derivative of $y = \dfrac{x^2\,\sqrt{e^x}}{8x+3}$

Take ln on both sides of the equation and use log properties:

$$\ln(y) = \ln\left[\frac{x^2\,\sqrt{e^x}}{8x+3}\right] = \ln(x^2) + \tfrac{1}{2}\ \ln(e^x) - \ln(8x+3)$$

$$\ln(y) = 2\ \ln(x) + \tfrac{1}{2}\,x - \ln(8x+3)$$

Use implicit differentiation and log formula derivatives:

$$\frac{1}{y}\,y' = 2\,\frac{1}{x} + \frac{1}{2} - \frac{8}{8x+3} \implies y' = y\left[\frac{2}{x} + \frac{1}{2} - \frac{8}{8x+3}\right] = \frac{x^2\,\sqrt{e^x}}{8x+3}\left[\frac{2}{x} + \frac{1}{2} - \frac{8}{8x+3}\right]$$