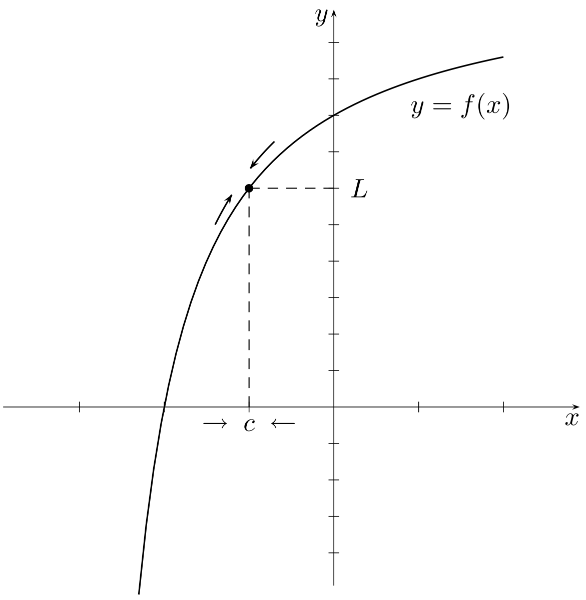
Limit Concept

Intuitive Definition :

- Given a function $f(x)$ as a graph or an equation, find y value as x approaches a value c.
- The concept of approaching value c from both sides of c is important. It can be done visually or by choosing values very close to the value c.
- The value on the left of c should be $c - 0.001$ or closer and the value on the right of c should be $c + 0.001$ or closer.
- If the limit of the function $f(x)$ on both sides of $x = c$ is the same value L, then the limit exists and its value is the y value L.
- If the limit on both sides is different, we say: no limit or limit does not exist (D.N.E.)

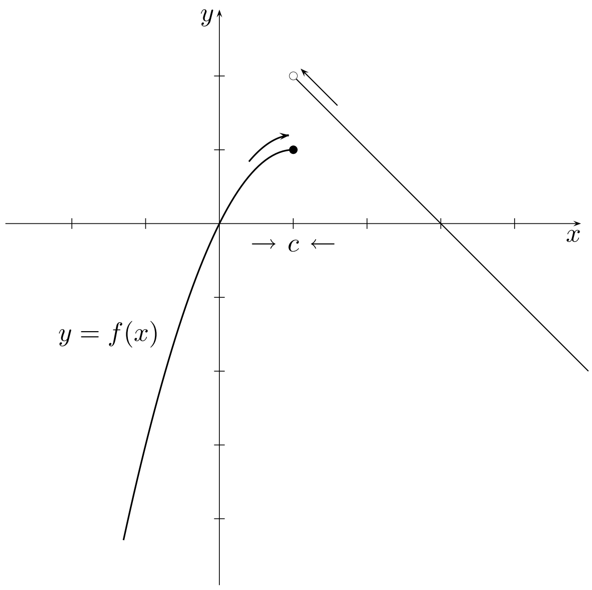
Graphical Limit

limit exists at $x = c$



$$\lim_{x \to c} f(x) = L$$

limit does not exist at $x = c$



$$\lim_{x \to c} f(x) = D.N.E.$$

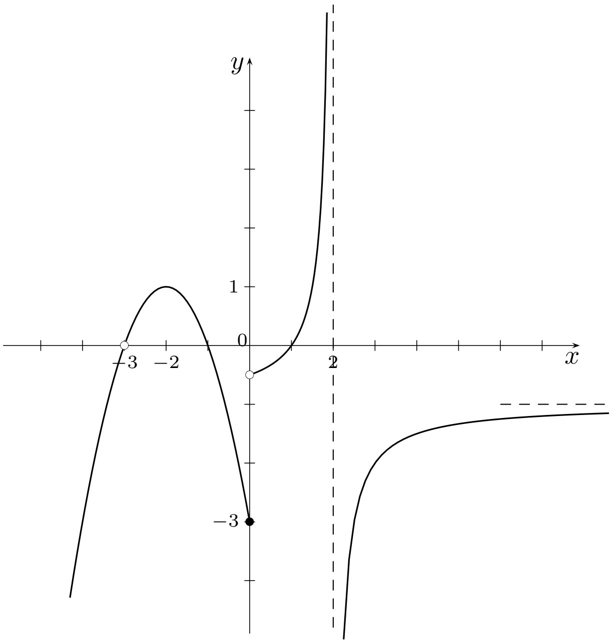
Limit Notation & Symbol :

A function $f(x)$ is given by a graph or an equation; x approaches a value c and we want to find if the y value exists or does not exist (no limit).

If the limit exists $\longrightarrow \lim_{x \to c} f(x) = L$; if no limit exists $\longrightarrow \lim_{x \to c} f(x) = D.N.E.$

Note: Pay attention to the presentation of the notation.

Example 1: Given the graph of a function $f(x)$, find:



(a) $\lim_{x\to 0} f(x)$

Examine left of $x=0$,
note:
minus sign is shown on top right of value 0
$\lim_{x\to 0^-} f(x)=-3$

Examine right of $x=0$,
note:
plus sign is shown on top right of value 0
$\lim_{x\to 0^+} f(x)=-\frac{1}{2}$

y values are not the same; therefore
$\lim_{x\to 0} f(x)=D.N.E.$

Note: point at $x=0$; $y=f(0)=-3$ (full dot) has coordinates $(0,-3)$

(b) $\lim_{x\to -3} f(x)$

Visually, y value to the left of $x=-3$ is same y value to the right of $x=-3$; so we say:

$\lim_{x\to -3} f(x)=0$

Note: at $x=-3$, point does not exist but the limit exits.

(c) $\lim_{x\to 2} f(x)$

Examine left of $x=2$,
note:
minus sign is shown on top right of value 2
$\lim_{x\to 2^-} f(x)=+\infty$

Examine right of $x=2$,
note:
plus sign is shown on top right of value 2
$\lim_{x\to 2^+} f(x)=-\infty$

y values are not the same; therefore
$\lim_{x\to 2} f(x)=D.N.E.$

Note: $f(x)$ is undefined at $x=2$ (vertical asymptote)

(d) $\lim_{x\to +\infty} f(x)$

The curve of $f(x)$ as x approaches infinity shows that y value approaches -1 ; so we say:

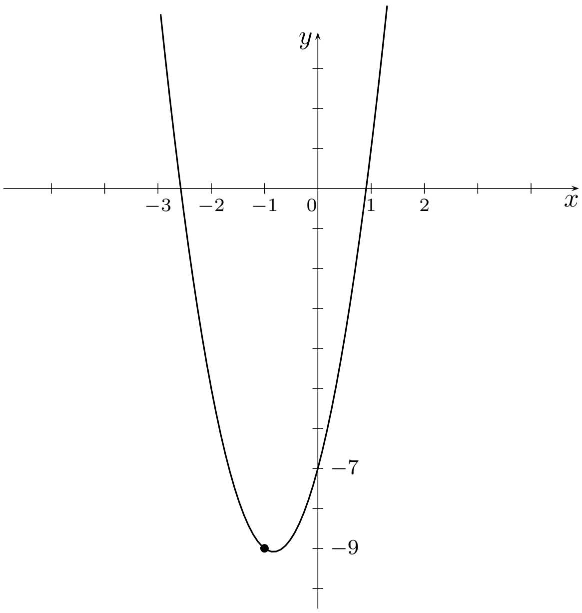
$\lim_{x\to +\infty} f(x)=-1$

Note: at x is approaching infinity, y approaches -1 (horizontal asymptote).

Example 2 (parabola): Given the equation of a function $y = f(x) = 3x^2 + 5x - 7$, find: $\lim\limits_{x \to -1} f(x)$

Since the function is a parabola, replace in $f(x)$ each x with -1 ; we get

$y = f(-1) = 3(-1)^2 + 5(-1) - 7 = -9$



Therefore: $\lim\limits_{x \to -1} f(x) = \lim\limits_{x \to -1} (3x^2 + 5x - 7)$

$\lim\limits_{x \to -1} f(x) = 3(-1)^2 + 5(-1) - 7 = -9$

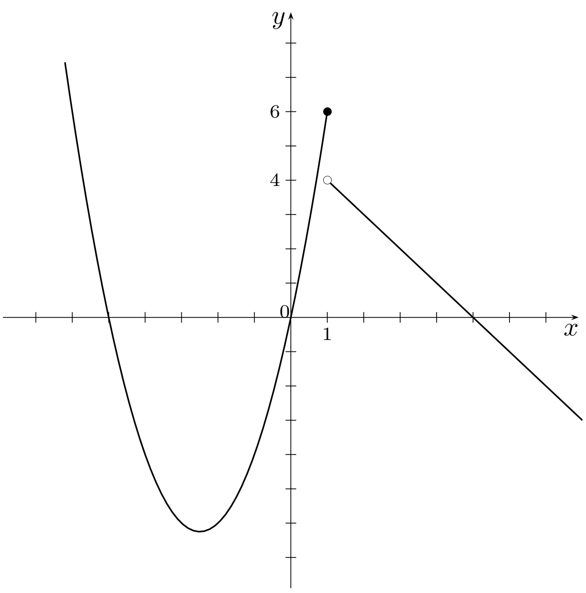
Note:
Pay attention to the presentation of the notation.

the graph shows that $\lim\limits_{x \to -1} f(x) = -9$

Note: The graph is not part of the solution.
It is just visual !

Example 3 (piecewise graph): Given the equation of a function $y = f(x) = \begin{cases} x^2 + 5x & \text{if} \quad x \leq 1 \\ -x + 4 & \text{if} \quad x > 1 \end{cases}$,

find: $\lim\limits_{x \to 1} f(x)$



since $f(x)$ is a piecewise :
at $x < 1$ (left side of $x = 1$)
use the upper equation;
at $x > 1$ (right side of $x = 1$)
use the lower equation.

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (x^2 + 5x) = 6$
$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (-x + 5) = 4$

$\lim\limits_{x \to 1^-} f(x) \neq \lim\limits_{x \to 1^+} f(x) \to \lim\limits_{x \to 1} f(x) = D.N.E.$

Note:
Pay attention to the presentation of the notation.

the graph shows that $\lim\limits_{x \to 1} f(x) = D.N.E.$

Notice: The point exists at $x = 1$ with
$y = f(1) = (1)2 + 5(1) = 6$
Point coordinates: $(1, 6)$
The point at $(1, 4)$ is an empty dot.

Note: The graph is not part of the solution.
It is just visual !

Example 4 (rational function): Given the equation of a function $y = f(x) = \dfrac{x^2 - 25}{x^2}$, find: $\lim\limits_{x \to 0} f(x)$

To test $\lim\limits_{x \to 0^-} f(x)$, use $x = -0.0001$

$$\frac{(-0.0001)^2 - 25}{(-0.0001)^2} = -24\,999\,999$$

$$\lim_{x \to 0^-} f(x) = -\infty$$

To test $\lim\limits_{x \to 0^+} f(x)$, use $x = 0.0001$

$$\frac{(0.0001)^2 - 25}{(0.0001)^2} = -24\,999\,999$$

$$\lim_{x \to 0^+} f(x) = -\infty$$

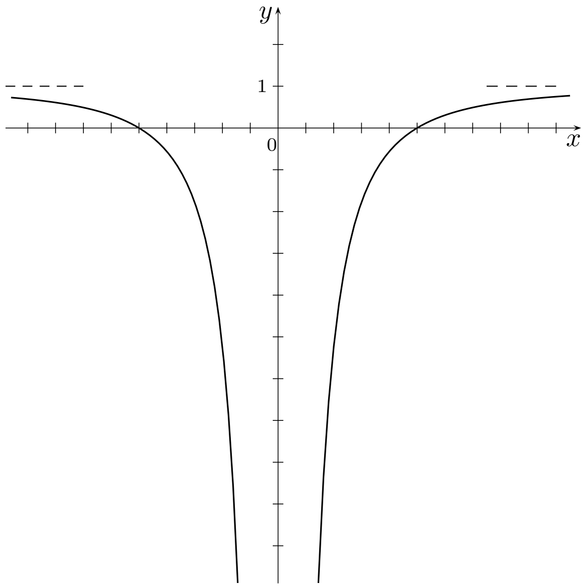
$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} \to f(x) = -\infty$$

Note:
limit of a function undefined at $x = 0$.

the graph shows that $\lim\limits_{x \to 0} f(x) = -\infty$

The curve of $f(x)$ shows that on both sides of $x = 0$, the y value goes to negative infinity.
Also observe that:

$$\lim_{x \to -\infty} f(x) = \lim_{x \to +\infty} f(x) = 1$$



Note: The graph is not part of the solution.
It is just visual !

Example 5 (limit to infinity): Given the equation of a function $y = f(x) = \dfrac{24 - 6x^2}{3x^2 + 4}$, find $\lim\limits_{x \to -\infty} f(x)$

Limit to infinity implies dominant terms of $f(x)$:
use term with highest power of x in numerator divided by term with highest power of x in denominator.

$\dfrac{-6x^2}{3x^2}$ is reduced to -2 so $\lim\limits_{x \to -\infty} f(x) = -2$

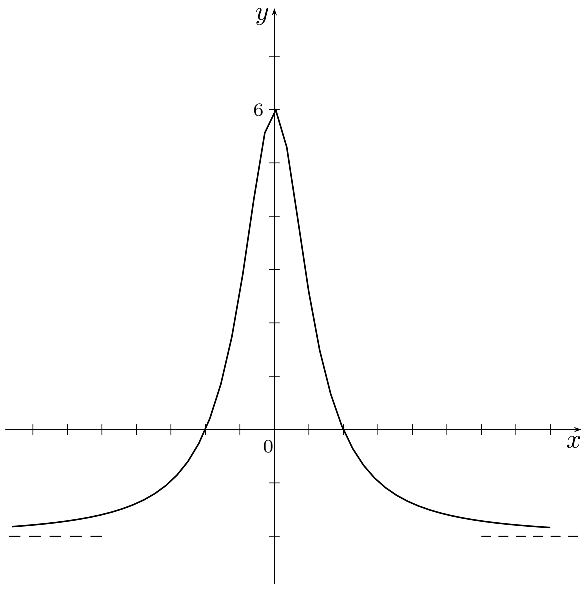
Note: if you replace x with a very big negative number in $f(x)$,
the y value will be 2 or very close to -2 .
the graph shows that $\lim\limits_{x \to -\infty} f(x) = -2$
Try it with your calculator.
Another way:

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{24 - 6x^2}{3x^2 + 4}$$

divide each term by x^2

$$\lim_{x \to -\infty} \frac{\dfrac{24}{x^2} - \dfrac{6x^2}{x^2}}{\dfrac{3x^2}{x^2} + \dfrac{4}{x^2}} \text{ then reduce:}$$

$$\lim_{x \to -\infty} \frac{\dfrac{24}{x^2} - 6}{3 + \dfrac{4}{x^2}} = \frac{0 - 6}{3 + 0} = -2$$

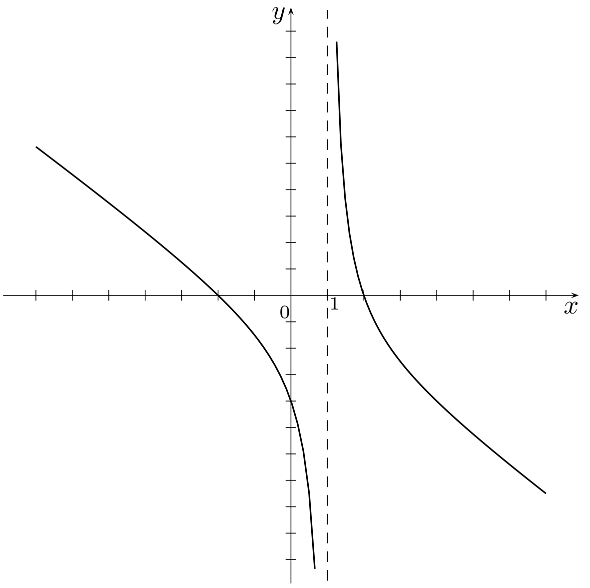


Note: The graph is not part of the solution.
It is just visual !

Example 6 (rational function): Given the equation of a function $y = f(x) = \dfrac{4 - x^2}{x - 1}$,

find: (a) $\lim_{x \to 1^-} f(x)$; (b) $\lim_{x \to 1^+} f(x)$; (c) $\lim_{x \to +\infty} f(x)$; (d) $\lim_{x \to -\infty} f(x)$

Note: The solution of each question must be shown without using a graph. The graph of $f(x)$ is displayed for a visual verification. It is not the solution.



(6 a) one sided limit: $\lim_{x \to 1^-} f(x)$
choose $x = 0.999$ and replace in $f(x)$:

$$\frac{4 - (0.999)^2}{(0.999) - 1} = -3002$$

choose $x = 0.9999$ and replace in $f(x)$:

$$\frac{4 - (0.9999)^2}{(0.9999) - 1} = -30002$$

The calculations show that the closer we choose from the left side of $x = 1$,
we get a bigger negative number.

Therefore $\lim_{x \to 1^-} f(x) = -\infty$

(6 b) one sided limit: $\lim_{x \to 1^+} f(x)$

choose $x = 1.001$ and replace in $f(x)$: $\dfrac{4 - (1.001)^2}{(1.001) - 1} = 2998$

choose $x = 1.0001$ and replace in $f(x)$: $\dfrac{4 - (1.0001)^2}{(1.0001) - 1} = 29998$

The calculations show that the closer we choose from the right side of $x = 1$,
we get a bigger positive number.

Therefore $\lim_{x \to 1^+} f(x) = +\infty$

(6 c) limit to infinity: $\lim_{x \to +\infty} f(x)$

Limit to infinity implies dominant terms of $f(x)$:
use term with highest power of x in numerator divided by term with highest power of x in denominator.

$\dfrac{-x^2}{x}$ is reduced to $-x$; when $x \to +\infty$, $y \to -\infty$; so $\lim_{x \to +\infty} f(x) = -\infty$

Note: if you replace x with a very big positive number in $f(x)$, the y value will be a very big negative number. Try it with your calculator.

(6 d) limit to infinity: $\lim_{x \to -\infty} f(x)$

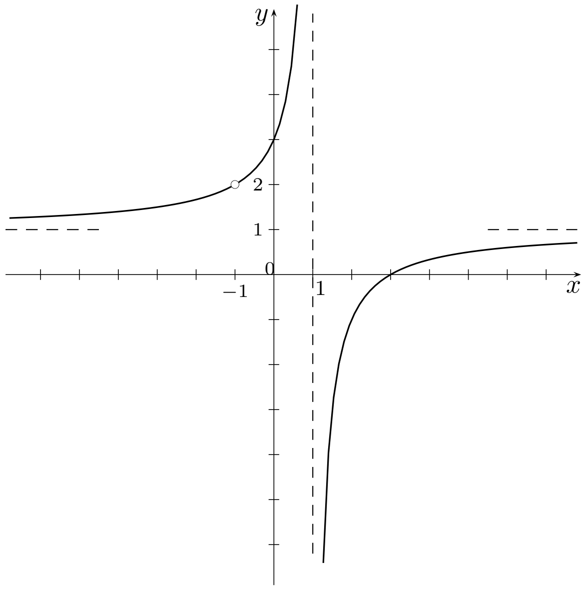
Limit to infinity implies dominant terms of $f(x)$:
use term with highest power of x in numerator divided by term with highest power of x in denominator.

$\dfrac{-x^2}{x}$ is reduced to $-x$; when $x \to -\infty$, $y \to +\infty$; so $\lim_{x \to -\infty} f(x) = +\infty$

Note: if you replace x with a very big negative number in $f(x)$, the y value will be a very big positive number. Try it with your calculator.

Example 7 $\left(\text{indeterminate form } \frac{0}{0}\right)$:

Given the equation of a function $y = f(x) = \dfrac{x^2 - 2x - 3}{x^2 - 1}$, find: $\lim\limits_{x \to -1} f(x)$



replace in $f(x)$, x with -1 , we get:

$$\frac{(-1)^2 - 2(-1) - 3}{(-1)^2 - 1} = \frac{0}{0} \text{ indeterminate form;}$$

use factoring:

$$\frac{(x+1)(x-3)}{(x+1)(x-1)} = \frac{x-3}{x-1}$$

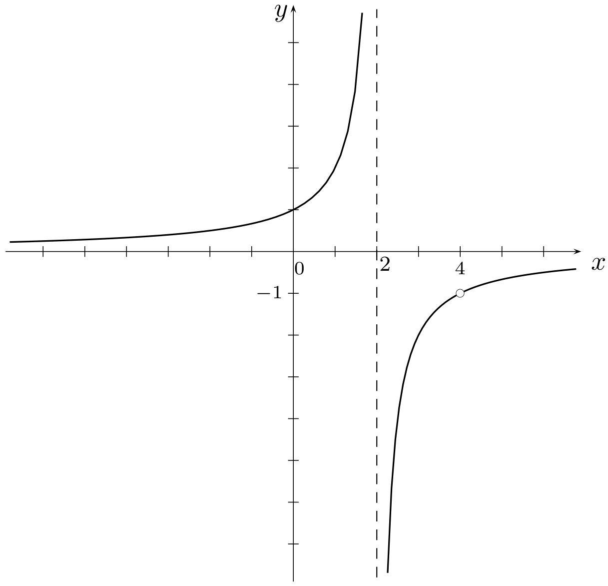
$$\lim_{x \to -1} \frac{x-3}{x-1} = \frac{(-1)-3}{(-1)-1} = 2$$

Note: Indeterminate form implies Algebra work to be done to remove the indetermination. Indeterminate is not the same as undefined !

the curve has an empty dot at point $(-1, 2)$. The empty dot indicates a break in the curve.
Note: The graph is not part of the solution. It is just visual !

Example 8 $\left(\text{indeterminate form } \frac{0}{0}\right)$:

Given the equation of a function $y = f(x) = \dfrac{\frac{4}{x-2} - 2}{x - 4}$, find: $\lim\limits_{x \to 4} f(x)$



replace in $f(x)$, x with 4 , we get:

$$\frac{\frac{4}{(4)-2} - 2}{(4) - 4} = \frac{0}{0} \text{ indeterminate form;}$$

common denominator and reducing:

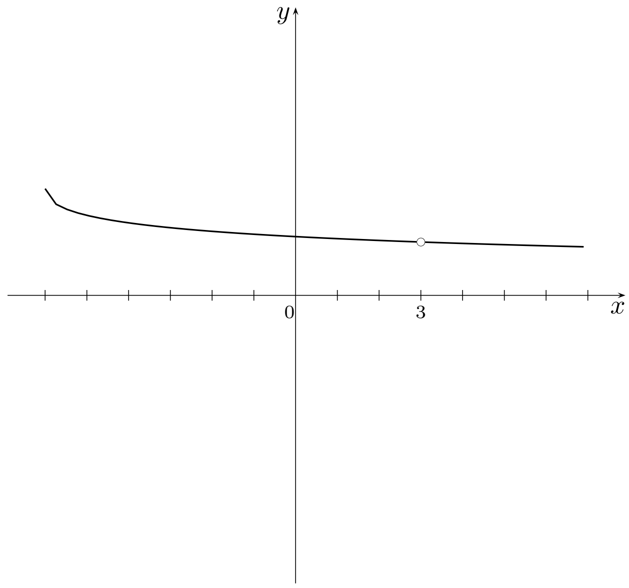
$$\frac{\frac{4-2(x-2)}{x-2}}{x-4} = \frac{\frac{8-2x}{x-2}}{x-4} = \frac{-2(x-4)}{x-2} \cdot \frac{1}{x-4}$$

$$= \frac{-2}{x-2} \implies \lim_{x \to 4} \frac{-2}{x-2} = -1$$

Note: Algebra work to be done to remove the indetermination is the common denominator to reduce fractions.

the curve has an empty dot at point $(4, -1)$. The point $(4, -1)$ does not exist on the curve.
The curve is the reduced function $f(x) = \dfrac{2}{2-x}$
Note: The graph is not part of the solution. It is just visual !

Example 9: Given the equation of a function $y = f(x) = \dfrac{\sqrt{x+6}-3}{x-3}$, find: $\lim\limits_{x\to 3} f(x)$



replace in $f(x)$, x with 3 , we get:

$\dfrac{\sqrt{(3)+6}-3}{(3)-3} = \dfrac{0}{0}$ indeterminate form;

rationalize numerator and reducing:

$$\frac{(\sqrt{x+6}-3)(\sqrt{x+6}+3)}{(x-3)(\sqrt{x+6}+3)}$$

$$= \frac{x+6-9}{(x-3)(\sqrt{x+6}+3)} = \frac{x-3}{(x-3)(\sqrt{x+6}+3)}$$

$$\lim_{x\to 3} \frac{1}{\sqrt{x+6}+3} = \tfrac{1}{6}$$

Note: Algebra work to be done to remove the indetermination is rationalize numerator to reduce radicals.

the curve has an empty dot at point $\left(3, \frac{1}{6}\right)$.
The point $\left(3, \frac{1}{6}\right)$ does not exist on the curve.
The curve is the reduced function

$$f(x) = \frac{1}{\sqrt{x+6}+3}$$

Note: The graph is not part of the solution. It is just visual !

Limit Properties :

Let a and k be real numbers; x is a variable ; p a positive integer

(1) $\lim\limits_{x\to a} (k) = k$

(2) $\lim\limits_{x\to a} (x) = a$

(3) $\lim\limits_{x\to a} (x^p) = a^p$

(4) $\lim\limits_{x\to a} \left(\sqrt[p]{x}\right) = \sqrt[p]{a}$

(5) $\lim\limits_{x\to a} \Big(k\, f(x)\Big) = k \lim\limits_{x\to a} f(x)$

(6) $\lim\limits_{x\to a} \Big(f(x) \pm g(x)\Big) = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x)$

(7) $\lim\limits_{x\to a} \Big(f(x) \,.\, g(x)\Big) = \lim\limits_{x\to a} f(x) \,.\, \lim\limits_{x\to a} g(x)$

(8) $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$

(9) $\lim\limits_{x\to a} \Big(f(x)\Big)^p = \Big(\lim\limits_{x\to a} f(x)\Big)^p$

(10) $\lim\limits_{x\to a} \sqrt[p]{f(x)} = \sqrt[p]{\lim\limits_{x\to a} f(x)}$