
Why Use the Rules ?

Based on the definition of the derivative, you notice that the computations and simplifications of the expressions are rather difficult and tedious.

Instead of using the definition of the derivative, rules have been developed to reduce the amount of work and to get the derivative faster.

Power Rule

In the examples shown in the tutorial: Definition of the Derivative, examine the following patterns:

The term has form of $k\,x^n$ where k is the coefficient and n is the exponent;
k and n are real numbers, x is the variable.

The examples shown in the tutorial: Definition of the derivative gave:

in example 1: $3x^2$ has derivative $6x$

in example 2: $\dfrac{5}{x} = 5x^{-1}$ has derivative $5x^{-2}$

in example 3: $4\sqrt{x} = 4x^{1/2}$ has derivative $2x^{-1/2}$

in example 4: $6 = 6x^0$ has derivative 0

The Power Rule: $\boxed{k\,x^n \text{ has derivative } n\,k\,x^{n-1}}$

Example 1: Given the function $y = 15x^4$,

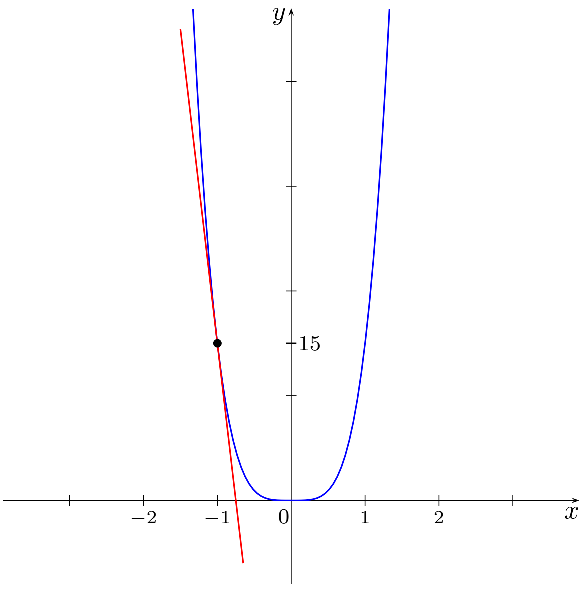
(a) Find the derivative

Use the power rule with form $k\,x^n$
$y\,' = 15(4)x^{4-1} = 60x^3$

(b) Evaluate the slope of the tangent line at $x = -1$
$y\,'(-1) = 60(-1)^3 = -60$

(c) Find the equation of the tangent line at $x = -1$

point at $x = -1$; $y = 15(-1)^4 = 15$ and slope $m = -60$
equation of tangent line: $y = -60x - 45$



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tangent line in red

Example 2: Given the function $y = \dfrac{20}{x^3} = 20x^{-3}$,

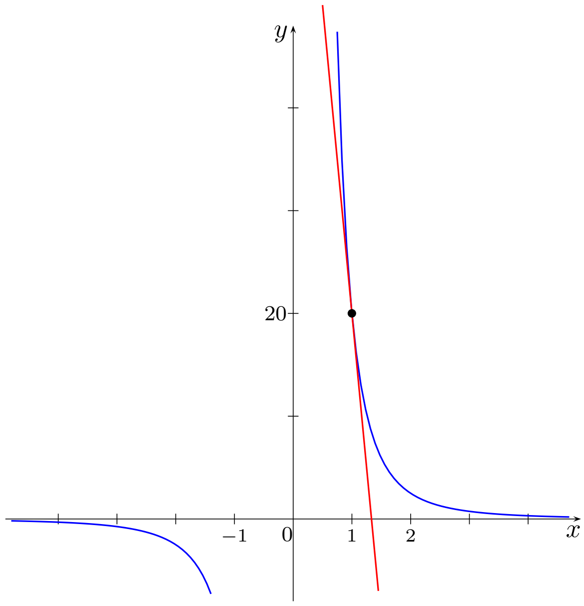
(a) Find the derivative

Use the power rule with form $k\,x^n$
$y' = 20(-3)x^{-3-1} = -60x^{-4}$

(b) Evaluate the slope of the tangent line at $x = 1$
$y'(1) = -60(1)^{-4} = -60$

(c) Find the equation of the tangent line at $x = 1$

point at $x = 1$; $y = \frac{20}{(1)^3} = 20$ and
slope $m = -60$
equation of tangent line: $y = -60x + 80$



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tangent line in red

Example 3: Given the function $y = \dfrac{10}{\sqrt{x}} = 10x^{-1/2}$,

(a) Find the derivative

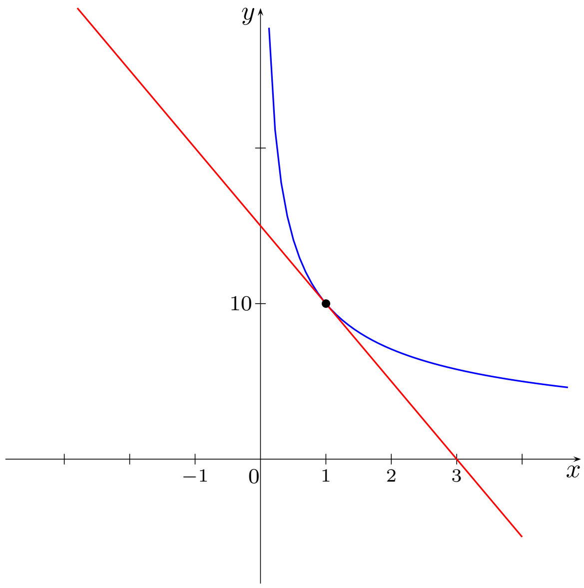
Use the power rule with form $k\,x^n$

$y' = 10\left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1} = -5x^{-3/2} = -\dfrac{5}{x\sqrt{x}}$

(b) Evaluate the slope of the tangent line at $x = 1$
$y'(1) = -5(1)^{-3/2} = -5$

(c) Find the equation of the tangent line at $x = 1$

point at $x = 1$; $y = \dfrac{10}{\sqrt{(1)}} = 10$ and
slope $m = -5$
equation of tangent line: $y = -5x + 15$



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tangent line in red

Example 4: Given the function $y = 20x^{1/4}$,

(a) Find the derivative

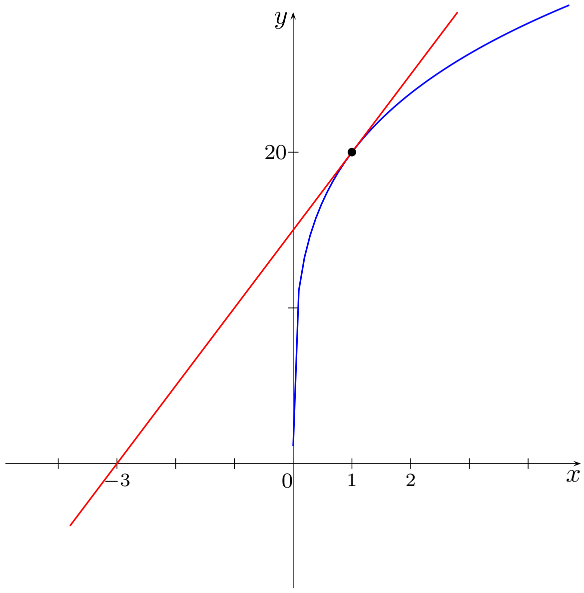
Use the power rule with form $k\,x^n$

$y\,' = 20\,\left(\frac{1}{4}\right)x^{\frac{1}{4}-1} = 5x^{-3/4}$

(b) Evaluate the slope of the tangent line at $x = 1$
$y\,'(1) = 5(1)^{-3/4} = 5$

(c) Find the equation of the tangent line at $x = 1$

point at $x = 1$; $y = 20(1)^{-3/4} = 20$ and
slope $m = 5$
equation of tangent line: $y = 5x + 15$



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tangent line in red

Example 5: Given the function $y = 16 = 16x^0$,

(a) Find the derivative $\rightarrow$ use the power rule with form $k\,x^n \rightarrow y\,' = 16\,(0)x^{0-1} = 0$

(b) Evaluate the slope of the tangent line at $x = -2 \rightarrow y\,'(-2) = 16$

(c) Find the equation of the tangent line at $x = -2 \rightarrow$ point at $(-2, 16)$ and slope $m = 0$
equation of tangent line: $y = 16$

Note: The function $y = 16$ and the tangent line at $(-2, 16)$ are both the same horizontal line.

Addition & Subtraction Rule

Given the function $f(x) + g(x)$

The derivative of the sum of two terms or two functions is $\dfrac{dy}{dx} = y' = f'(x) + g'(x)$

Given the function $f(x) - g(x)$

The derivative of the difference of two terms or two functions is $\dfrac{dy}{dx} = y' = f'(x) - g'(x)$

Example 6: Given the function $y = \frac{1}{4}x^8 + 5x^2$,

(a) Find the derivative

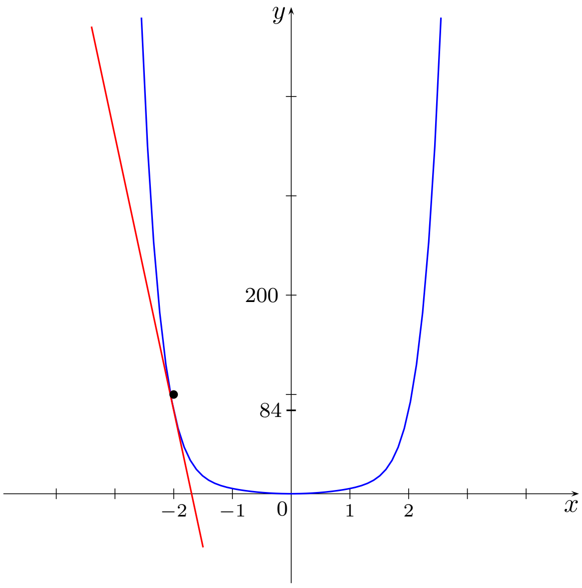
Use the power rule and the addition rule to find the derivative:

$$y' = \frac{1}{4}(8)x^{8-1} + 5\,(2)x^{2-1} = 2x^7 + 10x$$

(b) Evaluate the slope of the tangent line at $x = -2$
$y'(-2) = 2(-2)^7 + 10(-2) = -276$

(c) Find the equation of the tangent line at $x = -2$

point at $x = -2$; $y = \frac{1}{4}(-2)^8 + 5(-2)^2 = 84$ and slope $m = -276$
equation of tangent line: $y = -276x - 468$



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tangent line in red

Example 7: Given the function $y = 4x - \frac{1}{3}x^3$,

(a) Find the derivative

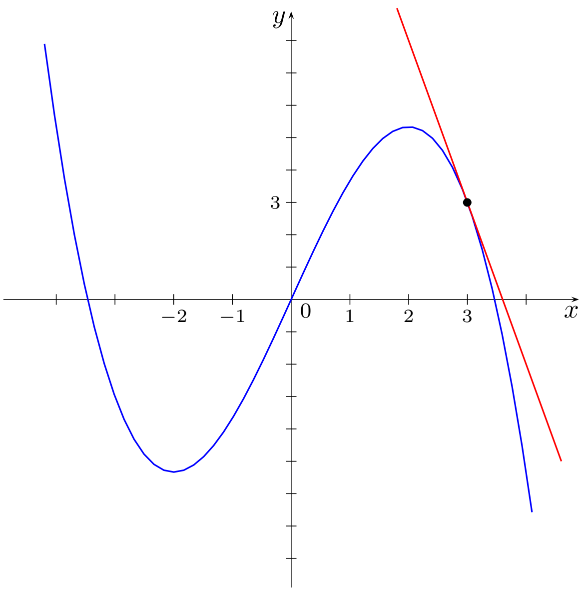
Use the power rule and the subtraction rule to find the derivative:

$$y' = 4(1)\,x^{1-1} - \frac{1}{3}(3)\,x^{3-1} = 4 - x^2$$

(b) Evaluate the slope of the tangent line at $x = 3$
$y'(3) = 4 - (3)^2 = -5$

(c) Find the equation of the tangent line at $x = 3$

point at $x = 3$; $y = 4(3) - \frac{1}{3}(3)^3 = 3$ and slope $m = -5$
equation of tangent line: $y = -5x + 18$



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Example 8: Given the function $y = 5 + 2x - x^2$,

(a) Find the derivative

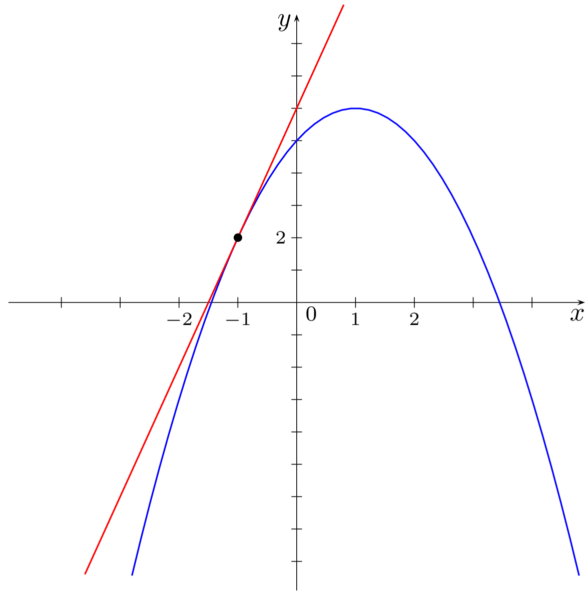
Use the power rule, the addition rule and the subtraction rule to find the derivative:

$y' = 5(0)\,x^{0-1} + 2(1)\,x^{1-1} - (2)\,x^{2-1} = 2 - 2x$

(b) Evaluate the slope of the tangent line at $x = -1$
$y'(-1) = 2 - 2(-1) = 4$

(c) Find the equation of the tangent line at $x = -1$

point at $x = -1$; $y = 5 + 2(-1) - (-1)^2 = 2$ and slope $m = 4$
equation of tangent line: $y = 4x + 6$



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Example 9: Given the function $y = 4x^3 - 4\sqrt{x} - \dfrac{2}{x} + 6x^{2/3} + 7$,

(a) Find the derivative

Use the power rule, the addition rule and the subtraction rule to find the derivative:

$$\frac{dy}{dx} = y' = 12x^2 - \tfrac{5}{2}\,x^{-1/2} + 2x^{-2} + 4x^{-1/3} = 12x^2 - \frac{5}{2\sqrt{x}} + \frac{2}{x^2} + \frac{4}{x^{1/3}}$$

(b) Evaluate the slope of the tangent line at $x = 1$

$$y'(1) = 12(1)^2 - \frac{5}{2\sqrt{(1)}} + \frac{2}{(1)^2} + \frac{4}{(1)^{1/3}} = \frac{31}{2} = 15.5$$

Product Rule

The function $y = u \, . \, v$ shows a product operation of 2 functions where u and v are functions of x .

To find the derivative, use product rule:

$$\boxed{\frac{dy}{dx} = y\,' = u' \, . \, v + v' \, . \, u}$$

Note: To use the product rule

- the multiplication operation must be in the function
- need u' and v' to replace them in the formula

Example 10: Given the function $y = (2x - 8) \, . \, (3x + 9) = 6x^2 - 6x - 72$,

(a) Find the derivative

Use the power rule and the product rule to find the derivative:

$u = 2x - 8 \rightarrow u' = 2$; $v = 3x + 9 \rightarrow v' = 3$

$\frac{dy}{dx} = y' = u' \, . \, v + v' \, . \, u$

$y\,' = 2(3x + 9) + 3(2x - 8) = 12 - 6x$

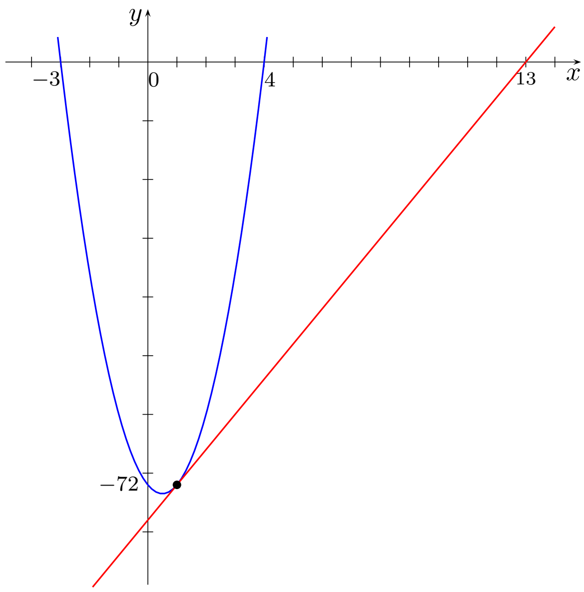
(b) Evaluate the slope of the tangent line at $x = 1$

$y\,'(1) = 12 - 6(1) = 6$

(c) Find the equation of the tangent line at $x = 1$

point at $x = 1$; $y = -72$ and slope $m = 6$

equation of tangent line: $y = 6x - 78$



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Example 11: Given the function $y = (x^5 - 6x^8 + 2) \, . \, (4 - x^3 + 5x^4)$,

(a) Find the derivative

Use the power rule and the product rule to find the derivative:

$u = x^5 - 6x^8 + 2 \rightarrow u' = 5x^4 - 48x^7 \;\; ; \;\; v = 4 - x^3 + 5x^4 \rightarrow v' = -3x^2 + 20x^3$

$\dfrac{dy}{dx} = y' = u' \, . \, v + v' \, . \, u$

$y' = (5x^4 - 48x^7)(4 - x^3 + 5x^4) + (-3x^2 + 20x^3)(x^5 - 6x^8 + 2)$

(b) Evaluate the slope of the tangent line at $x = 1$

$$y'(1) = \Big(5(1)^4 - 48(1)^7\Big)\Big(4 - (1)^3 + 5(1)^4\Big) + \Big(-3(1)^2 + 20(1)^3\Big)\Big((1)^5 - 6(1)^8 + 2\Big) = -395$$

Quotient Rule

The function $y = \dfrac{u}{v}$ shows a division operation of 2 functions where u and v are functions of x .

To find the derivative, use quotient rule:

$$\boxed{\frac{dy}{dx} = y' = \frac{u' \, . \, v - v' \, . \, u}{v^2}}$$

Note: To use the quotient rule

- the division operation must be in the function
- need u' and v' to replace them in the formula

Example 12: Given the function $y = \dfrac{x-1}{x^2}$,

(a) Find the derivative

Use the power rule and the quotient rule to find the derivative:

$u = x - 1 \rightarrow u' = 1$; $v = x^2 \rightarrow v' = 2x$

$\dfrac{dy}{dx} = y' = \dfrac{u' \,.\, v - v' \,.\, u}{v^2}$

$y\,' = \dfrac{(1)x^2 - (2x)(x-1)}{x^4} = \dfrac{-x+2}{x^3}$

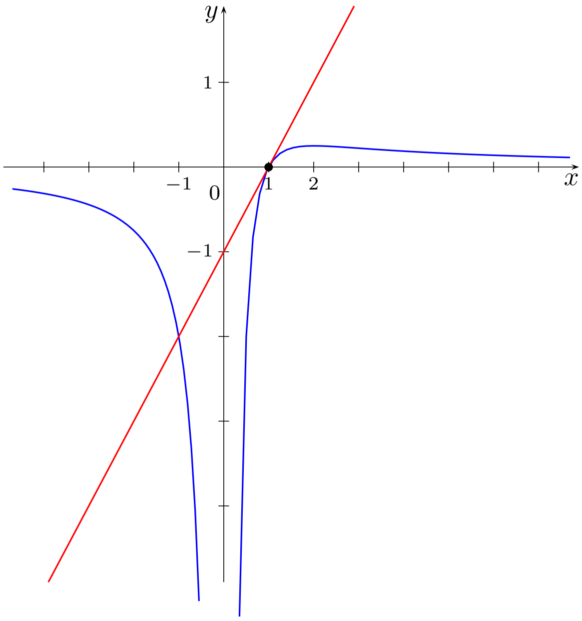
(b) Evaluate the slope of the tangent line at $x = 1$

$y\,'(1) = \dfrac{-(1)+2}{(1)^3} = 1$

(c) Find the equation of the tangent line at $x = 1$

point at $x = 1$; $y = 0$ and
slope $m = 1$

equation of tangent line: $y = x - 1$



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Example 13: Given the function $y = \dfrac{x^2}{2x+1}$,

(a) Find the derivative

Use the power rule and the quotient rule to find the derivative:

$u = x^2 \rightarrow u' = 2x$; $v = 2x + 1 \rightarrow v' = 2$

$\dfrac{dy}{dx} = y' = \dfrac{u' \,.\, v - v' \,.\, u}{v^2}$

$y\,' = \dfrac{(2x)(2x+1) - (2)x^2}{(2x+1)^2} = \dfrac{2x^2 + 2x}{(2x+1)^2}$

(b) Evaluate the slope of the tangent line at $x = 1$

$y\,'(1) = \dfrac{2(1)^2 + 2(1)}{\Big(2(1)+1\Big)^2} = \dfrac{4}{9} \approx 0.44$

<u>Chain Rule</u>

The function $y = u\Big(v(x)\Big) = (u \circ v)(x)$ shows a composition operation of 2 functions where u and v are functions of x.

To find the derivative, use chain rule:

$$\boxed{\frac{dy}{dx} = y' = u'\Big(v(x)\Big)\,.\,v'(x)}$$

General Power Rule: $\boxed{y = u^n \implies y' = n\,u^{n-1}\,.\,u'}$

<u>Note:</u> To use the chain rule

- the composition operation must be in the function
- need $u'(v)$ and v' to replace them in the formula

Example 14: Given the function $y = (4 - x^2)^3$,

(a) Find the derivative

Use the power rule and the chain rule to find the derivative:

$u\Big(v(x)\Big) = (4 - x^2)^3 \to u'\Big(v(x)\Big) = 3(4 - x^2)^2$

$v(x) = 4 - x^2 \to v'(x) = -2x$

$\frac{dy}{dx} = y' = u'\Big(v(x)\Big)\,.\,v'(x)$

$y' = 3(4 - x^2)^2\,.\,(-2x) = -6x\,(4 - x^2)^2$

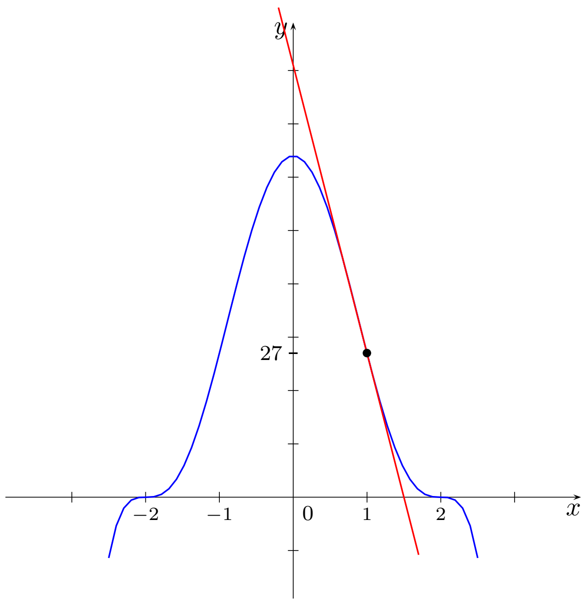
(b) Evaluate the slope of the tangent line at $x = 1$

$y'(1) = -6(1)\Big(4 - (1)^2\Big)^2 = -54$

(c) Find the equation of the tangent line at $x = 1$

point at $x = 1$; $y = 27$ and slope $m = -54$

equation of tangent line: $y = -54x + 81$



the curve of the function in blue; tangent line in red

Example 15: Given the function $y = (x^2 - 4x + 5)^7$,

(a) Find the derivative

Use the power rule and the chain rule to find the derivative:

$u\Big(v(x)\Big) = (x^2 - 4x + 5)^7 \rightarrow u'\Big(v(x)\Big) = 7(x^2 - 4x + 5)^6$

$v(x) = x^2 - 4x + 5 \rightarrow v'(x) = 2x - 4$

$\dfrac{dy}{dx} = y' = u'\Big(v(x)\Big) \,.\, v'(x)$

$y' = 7(x^2 - 4x + 5)^6 \,.\, (2x - 4) = 7(x^2 - 4x + 5)^6\,(2x - 4)$

(b) Evaluate the slope of the tangent line at $x = 1$

$y'(1) = 7\Big((1)^2 - 4(1) + 5\Big)^6\Big(2(1) - 4\Big) = -896$

Example 16: Given the function $y = (2x - 3)^4\,(5x + 1)^3$, find the derivative and factor the answer.

Use the product rule, the chain rule and the power rule to find the derivative:

$u = (2x - 3)^4 \rightarrow u' = 4(2x - 3)^3\,(2) = 8(2x - 3)^3$

$v = (5x + 1)^3 \rightarrow v' = 3(5x + 1)^2\,(5) = 15(5x + 1)^2$

$\dfrac{dy}{dx} = y' = u' \,.\, v + v' \,.\, u$

$y' = 8(2x - 3)^3\,(5x + 1)^3 + 15(5x + 1)^2\,(2x - 3)^4 = (2x - 3)^3\,(5x + 1)^2\,(70x - 37)$

Example 17: Given the function $y = \dfrac{4x}{(x-1)^2}$, find the derivative and simplify the answer.

Use the quotient rule, the chain rule and the power rule to find the derivative:

$u = 4x \rightarrow u' = 4 \;\;;\;\; v = (x - 1)^2 \rightarrow v' = 2(x - 1)$

$\dfrac{dy}{dx} = y' = \dfrac{u' \,.\, v - v' \,.\, u}{v^2}$

$y' = \dfrac{(4)(x-1)^2 - 2(x-1)(4x)}{(x-1)^4} = -\dfrac{4(x+1)}{(x-1)^3}$

Example 18 (derivatives rules): Given $f(1) = -2$, $f'(1) = -5$ and $g(x) = 3x^2 \,.\, f(x)$, find $g'(1)$.

Use the product rule to find the derivative:

$u = 3x^2 \to u' = 6x$; $v = f(x) \to v' = f'(x)$

$\dfrac{dy}{dx} = y' = u' \,.\, v + v' \,.\, u$

$y' = 6x \,.\, f(x) + f'(x) \,.\, 3x^2$, replace $x = 1 \implies g'(1) = 6(1) \,.\, f(1) + f'(1) \,.\, 3(1)^2 = 6(-2) + (-5)3 = -27$

Example 19 (derivatives rules): Given $f(2) = -4$, $f'(2) = 3$ and $g(x) = \dfrac{f(x)}{2x-3}$, find $g'(2)$.

Use the quotient rule to find the derivative:

$u = f(x) \to u' = f'(x)$; $v = 2x - 3 \to v' = 2$

$\dfrac{dy}{dx} = y' = \dfrac{u' \,.\, v - v' \,.\, u}{v^2}$

$y' = \dfrac{f'(x)\,(2x-3) - (2)\,f(x)}{(2x-3)^2}$, replace $x = 2 \implies g'(2) = \dfrac{f'(2)\,(4-3) - (2)\,f(2)}{(4-3)^2} = \dfrac{(3)(1) - (2)(-4)}{1} = 11$

Higher Order Derivatives

Given the function $y = f(x)$, the notation of the first order derivative: $y' = \dfrac{dy}{dx} = f'(x) = \dfrac{df}{dx}$

The derivative of the first order derivative is the second order derivative: $y'' = \dfrac{d^2y}{dx^2} = f''(x) = \dfrac{d^2f}{dx^2}$

and so on, ...

The derivative of the second order derivative is the third order derivative: $y^{(3)} = \dfrac{d^3y}{dx^3} = f^{(3)}(x) = \dfrac{d^3f}{dx^3}$

Fourth order derivative: $y^{(4)} = \dfrac{d^4y}{dx^4} = f^{(4)}(x) = \dfrac{d^4f}{dx^4}$

Note: $y^{(6)}$ is the sixth order derivative; y^6 is the sixth power of y . Do not be confuse!

Example 20: Given $y = \dfrac{2}{x^3} = 2x^{-3}$, find $\left.\dfrac{d^3y}{dx^3}\right|_{x=-2}$

Use the Power Rule: $\dfrac{dy}{dx} = 6x^{-4} \longrightarrow \dfrac{d^2y}{dx^2} = -24x^{-5} \longrightarrow \dfrac{d^3y}{dx^3} = -120x^{-6} = -\dfrac{120}{x^6}$

replace $x = -2 \longrightarrow \left.\dfrac{d^3y}{dx^3}\right|_{x=-2} = -\dfrac{120}{(-2)^6} = -\dfrac{15}{8}$

Example 21: Given $f(x) = (2+3x)^{7/3}$, find $f^{(3)}(2)$

Use the Chain Rule: $f'(x) = \frac{7}{3}(2+3x)^{4/3}(3) = 7(2+3x)^{4/3} \longrightarrow f''(x) = 7\left(\frac{4}{3}\right)(2+3x)^{1/3}(3) = 28(2+3x)^{1/3}$

$f^{(3)}(x) = 28\left(\frac{1}{3}\right)(2+3x)^{-2/3}(3) = 28(2+3x)^{-2/3} = \dfrac{28}{(2+3x)^{2/3}}$

replace $x = 2 \longrightarrow f^{(3)}(2) = \dfrac{28}{(2+6)^{2/3}} = \dfrac{28}{4} = 7$

Example 22: Given $y = e^{2x+6}$, find $y^{(4)}(-3)$

Use the Chain Rule: $y' = 2\,e^{2x+6} \longrightarrow y'' = 4\,e^{2x+6} \longrightarrow y^{(3)} = 8\,e^{2x+6} \longrightarrow y^{(4)} = 16\,e^{2x+6}$

replace $x = -3 \longrightarrow y^{(4)}(-3) = 16\,e^{-6+6} = 16$

Example 23: Given $y = \sin(3x)$, find $\left.\dfrac{d^3y}{dx^3}\right|_{x=2\pi}$

Use the chain Rule: $\dfrac{dy}{dx} = 3\,\cos(3x) \longrightarrow \dfrac{d^2y}{dx^2} = -9\,\sin(3x) \longrightarrow \dfrac{d^3y}{dx^3} = -27\,\cos(3x)$

replace $x = 2\pi \longrightarrow \left.\dfrac{d^3y}{dx^3}\right|_{x=2\pi} = -27\,\cos(6\pi) = -27$