Topics of Trigonometry used in Calculus

Trigonometric Ratios in right triangle

Given a right triangle ABC with sides a , b and c ; with angles α , β and γ as shown.

The sides a , b and c are measured in units of length.
The angles α , β and γ are measured in degrees, radians or multiples of π .

The six trigonometric ratios for angle α are:

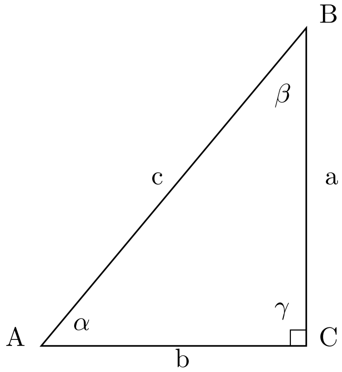
$\sin\ \alpha = \dfrac{a}{c}$; $\cos\ \alpha = \dfrac{b}{c}$; $\tan\ \alpha = \dfrac{a}{b}$

$\csc\ \alpha = \dfrac{c}{a}$; $\sec\ \alpha = \dfrac{c}{b}$; $\cot\ \alpha = \dfrac{b}{a}$

By definition:

$\tan\ \alpha = \dfrac{\sin\ \alpha}{\cos\ \alpha}$; $\csc\ \alpha = \dfrac{1}{\sin\ \alpha}$

$\sec\ \alpha = \dfrac{1}{\cos\ \alpha}$; $\cot\ \alpha = \dfrac{\cos\ \alpha}{\sin\ \alpha} = \dfrac{1}{\tan\ \alpha}$



Definition of Trigonometric ratios

The definition of Trigonometric ratios for angle α are:

sine of angle $\alpha = \dfrac{\text{opposite side to angle } \alpha}{\text{hypotenuse}}$

cosine of angle $\alpha = \dfrac{\text{adjacent side to angle } \alpha}{\text{hypotenuse}}$

cosecant of angle $\alpha = \dfrac{\text{hypotenuse}}{\text{opposite side to angle } \alpha}$

secant of angle $\alpha = \dfrac{\text{hypotenuse}}{\text{adjacent side to angle } \alpha}$

tangent of angle $\alpha = \dfrac{\text{opposite side to angle } \alpha}{\text{adjacent side to angle } \alpha}$

cotangent of angle $\alpha = \dfrac{\text{adjacent side to angle } \alpha}{\text{opposite side to angle } \alpha}$

Note: These ratios have no units.

Pythagorian Trigonometric identities

By Pythagorian Theorem, we have:

$$a^2 + b^2 = c^2 \longrightarrow \frac{a^2}{c^2} + \frac{b^2}{c^2} = 1 \longrightarrow \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$

use the trigonometric ratios to get the following Pythagorian trigonometric identities:

$$(\sin\ \alpha)^2 + (\cos\ \alpha)^2 = 1 \longrightarrow \sin^2\ \alpha + \cos^2\ \alpha = 1$$

$$\frac{\sin^2\ \alpha}{\cos^2\ \alpha} + \frac{\cos^2\ \alpha}{\cos^2\ \alpha} = \frac{1}{\cos^2\ \alpha} \longrightarrow \tan^2\ \alpha + 1 = \sec^2\ \alpha$$

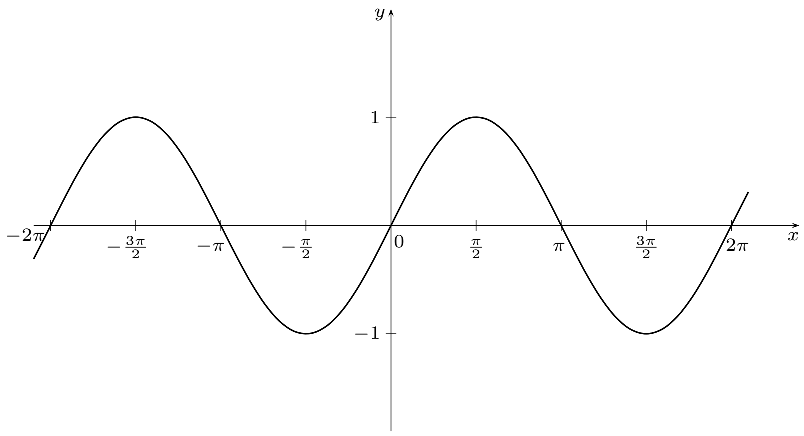
$$\frac{\sin^2\ \alpha}{\sin^2\ \alpha} + \frac{\cos^2\ \alpha}{\sin^2\ \alpha} = \frac{1}{\sin^2\ \alpha} \longrightarrow 1 + \cot^2\ \alpha = \csc^2\ \alpha$$

Trigonometric Functions

The six trigonometric functions are shown below.

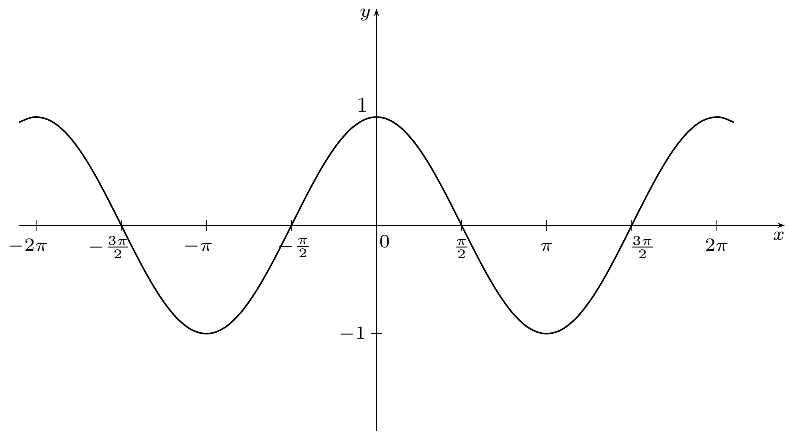
Each function will have its graph and main features like domain, range, asymptotes;
scale of horizontal axis will be in multiples of π (in radians) and the scale of vertical axis in units.
Each graph will show only one cycle counterclockwise (+) positive angles and one cycle clockwise (−) negative angles.
It does not imply that the graph ends.
The intercepts shown in the graph are not the only ones.
Many more intercepts will be described in a general way, the same for asymptotes not seen in the window of the graph.
The trigonometric functions are: $y = \sin\ x$; $y = \cos\ x$; $y = \tan\ x$; $y = \csc\ x$; $y = \sec\ x$; $y = \cot\ x$

The sine function



$y = \sin\ x$; domain: $\Re$; range: $[-1, 1]$; no asymptotes ; period: 2π ;
$x-$intercepts: $\ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$ in general: $n\,\pi$ where n is an integer ; $y-$intercept: 0

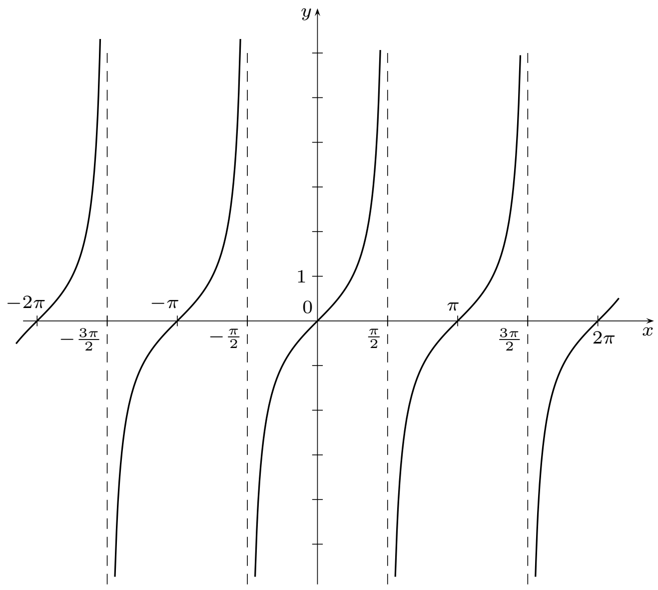
The cosine function



$y = \cos\ x$; domain: $\Re$; range: $[-1, 1]$; no asymptotes ; period: 2π ;

$x-$intercepts: $\ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$ in general: $\dfrac{(2n+1)\,\pi}{2}$ where n is an integer ; $y-$intercept: 1

The tangent function

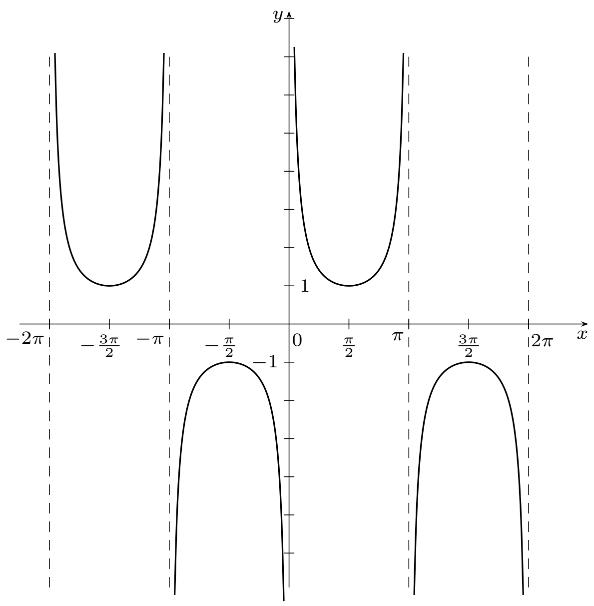


$y = \tan\ x$; domain: $\Re - \left\{ \dfrac{(2n+1)\,\pi}{2} \right\}$ where n is an integer ; range: $\Re$; period: π ;

vertical asymptotes at $x = \dfrac{(2n+1)\,\pi}{2}$ where n is an integer

$x-$intercepts: $\ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$ in general: $n\,\pi$ where n is an integer ; $y-$intercept: 0

The cosecant function

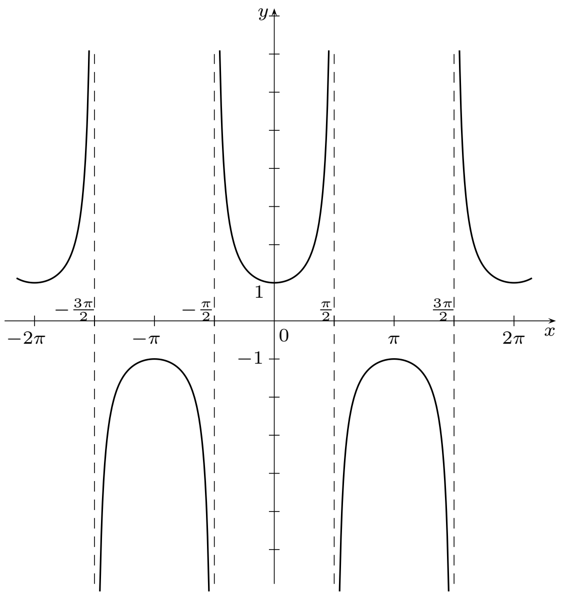


$y = \csc\ x$; domain: $\Re - \{n\,\pi\}$ where n is an integer ; range: $-1 \geq y \geq 1$; period: 2π ;

vertical asymptotes at $x = n\,\pi$ where n is an integer

no $x-$intercept ; no $y-$intercept

The secant function

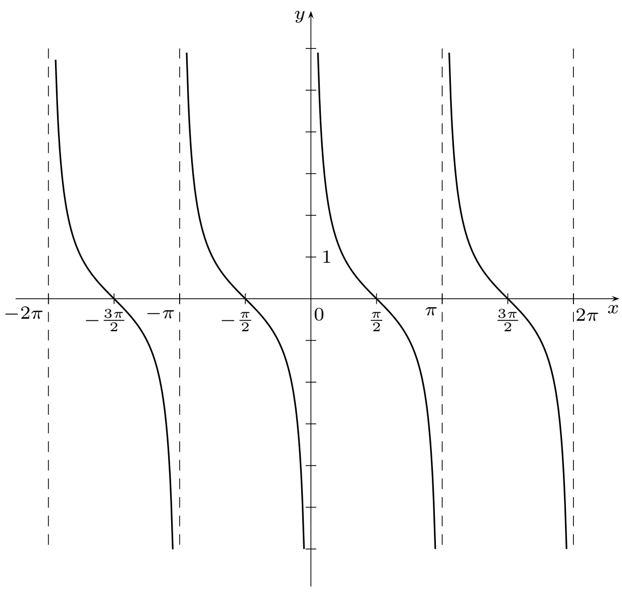


$y = \sec\ x$; domain: $\Re - \left\{ \frac{(2n+1)\,\pi}{2} \right\}$ where n is an integer ; range: $-1 \geq y \geq 1$; period: 2π ;

vertical asymptotes at $x = \frac{(2n+1)\,\pi}{2}$ where n is an integer

no $x-$intercept ; $y-$intercept: 1

The cotangent function



$y = \cot\ x$; domain: $\Re - \{n\,\pi\}$ where n is an integer ; range: $\Re$; period: π ;

vertical asymptotes at $x = n\,\pi$ where n is an integer

$x-$intercepts: $\ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$ in general: $\frac{(2n+1)\,\pi}{2}$ where n is an integer ; no $y-$intercept

Convert Degrees & Radians

To convert degrees (deg) to radians (rad) : (deg) $. \dfrac{\pi}{180°}$

Example: Convert 120° in radians $\longrightarrow 120° . \dfrac{\pi}{180°} = \dfrac{2\pi}{3}$ rad

To convert radians (rad) to degrees (deg) : (rad) $. \dfrac{180°}{\pi}$

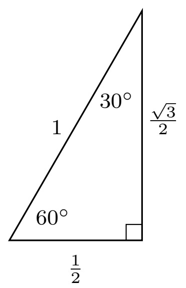
Example: Convert $\dfrac{4\pi}{3}$ rad in degrees $\longrightarrow \dfrac{4\pi}{3} . \dfrac{180°}{\pi} = 240°$

60-30 Right triangle Trigonometric values

Given a right triangle ABC with sides 1 , $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$; with angles $90° \left(\frac{\pi}{2}\right)$, $60° \left(\frac{\pi}{3}\right)$ and $30° \left(\frac{\pi}{6}\right)$ as shown.

$\sin 60° = \sin\left(\frac{\pi}{3}\right) = \cos 30° = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \approx 0.866$

$\cos 60° = \cos\left(\frac{\pi}{3}\right) = \sin 30° = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2} = 0.5$



$\tan 60° = \tan\left(\frac{\pi}{3}\right) = \cot 30° = \cot\left(\frac{\pi}{6}\right) = \sqrt{3} \approx 1.73$

$\cot 60° = \cot\left(\frac{\pi}{3}\right) = \tan 30° = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \approx 0.577$

$\csc 60° = \csc\left(\frac{\pi}{3}\right) = \sec 30° = \sec\left(\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \approx 1.15$

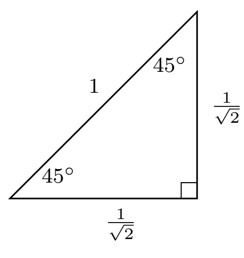
$\sec 60° = \sec\left(\frac{\pi}{3}\right) = \csc 30° = \csc\left(\frac{\pi}{6}\right) = 2$

45-45 Right triangle Trigonometric values

Given a right triangle ABC with sides 1 , $\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$; with angles $90° \left(\frac{\pi}{2}\right)$, $45° \left(\frac{\pi}{4}\right)$ and $45° \left(\frac{\pi}{4}\right)$ as shown.

$\sin 45° = \sin\left(\frac{\pi}{4}\right) = \cos 45° = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \approx 0.707$

$\cos 45° = \cos\left(\frac{\pi}{4}\right) = \sin 45° = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \approx 0.707$



$\tan 45° = \tan\left(\frac{\pi}{4}\right) = \cot 45° = \cot\left(\frac{\pi}{4}\right) = 1$

$\cot 45° = \cot\left(\frac{\pi}{4}\right) = \tan 45° = \tan\left(\frac{\pi}{4}\right) = 1$

$\csc 45° = \csc\left(\frac{\pi}{4}\right) = \sec 45° = \sec\left(\frac{\pi}{4}\right) = \sqrt{2} \approx 1.41$

$\sec 45° = \sec\left(\frac{\pi}{4}\right) = \csc 45° = \csc\left(\frac{\pi}{4}\right) = \sqrt{2} \approx 1.41$

degrees counterclockwise	radians	degrees clockwise	radians	sine	cosine	tangent	cosecant	secant	cotangent
$0°$	0	$-360°$	-2π	0	1	0	undefined	1	undefined
$30°$	$\frac{\pi}{6}$	$-330°$	$-\frac{11\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$	$\sqrt{3}$
$45°$	$\frac{\pi}{4}$	$-315°$	$-\frac{7\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$60°$	$\frac{\pi}{3}$	$-300°$	$-\frac{5\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2	$\frac{1}{\sqrt{3}}$
$90°$	$\frac{\pi}{2}$	$-270°$	$-\frac{3\pi}{2}$	1	0	undefined	1	undefined	0
$120°$	$\frac{2\pi}{3}$	$-240°$	$-\frac{4\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$\frac{2}{\sqrt{3}}$	-2	$-\frac{1}{\sqrt{3}}$
$135°$	$\frac{3\pi}{4}$	$-225°$	$-\frac{5\pi}{4}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	-1	$\sqrt{2}$	$-\sqrt{2}$	-1
$150°$	$\frac{5\pi}{6}$	$-210°$	$-\frac{7\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$	2	$-\frac{2}{\sqrt{3}}$	$-\sqrt{3}$
$180°$	π	$-180°$	$-\pi$	0	-1	0	undefined	-1	undefined
$210°$	$\frac{7\pi}{6}$	$-150°$	$-\frac{5\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	-2	$-\frac{2}{\sqrt{3}}$	$\sqrt{3}$
$225°$	$\frac{5\pi}{4}$	$-135°$	$-\frac{3\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	1	$-\sqrt{2}$	$-\sqrt{2}$	1
$240°$	$\frac{4\pi}{3}$	$-120°$	$-\frac{2\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$	$-\frac{2}{\sqrt{3}}$	-2	$\frac{1}{\sqrt{3}}$
$270°$	$\frac{3\pi}{2}$	$-90°$	$-\frac{\pi}{2}$	-1	0	undefined	-1	undefined	0
$300°$	$\frac{5\pi}{3}$	$-60°$	$-\frac{\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{2}{\sqrt{3}}$	2	$-\frac{1}{\sqrt{3}}$
$315°$	$\frac{7\pi}{4}$	$-45°$	$-\frac{\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	-1	$-\sqrt{2}$	$\sqrt{2}$	-1
$330°$	$\frac{11\pi}{6}$	$-30°$	$-\frac{\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$	-2	$\frac{2}{\sqrt{3}}$	$-\sqrt{3}$
$360°$	2π	$0°$	0	0	1	0	undefined	1	undefined

Useful Trigonometric Identities

The following trigonometric identities are helpful:

$\sin(-x) = -\sin(x)$

$\cos(-x) = \cos(x)$

$\sin(2x) = 2\,\sin(x)\,\cos(x)$

$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\,\cos^2(x) - 1 = 1 - 2\,\sin^2(x)$

$\sin(x+h) = \sin(x)\,\cos(h) + \sin(h)\,\cos(x)$

$\cos(x+h) = \cos(x)\,\cos(h) - \sin(x)\,sin(h)$

Examples Rewrite trigonometric expression

(a) $\dfrac{1}{\cos\,x+1} = \dfrac{1-\cos\,x}{(1+\cos\,x)(1-\cos\,x)} = \dfrac{1-\cos\,x}{1-\cos^2\,x} = \dfrac{1-\cos\,x}{\sin^2\,x} = \dfrac{1}{\sin^2\,x} - \dfrac{\cos\,x}{\sin\,x\,\sin\,x}$

$$= \frac{1}{(\sin\,x)^2} - \frac{\cos\,x}{\sin\,x\,\sin\,x} = \frac{1}{(\sin\,x)^2} - \frac{1}{\sin\,x} \,.\, \frac{\cos\,x}{\sin\,x} = \csc^2\,x - \csc\,x\,.\,\cot\,x = \csc\,x\,(\csc\,x - \cot\,x)$$

(b) $\dfrac{\cos\,2x}{\cos\,x} = \dfrac{\cos^2\,x - \sin^2\,x}{\cos\,x} = \dfrac{\cos\,x\,\cos\,x}{\cos\,x} - \dfrac{\sin\,x\,\sin\,x}{\cos\,x} = \cos\,x - \sin\,x\,\tan\,x$

OR

$$\frac{\cos\,2x}{\cos\,x} = \frac{2\cos^2\,x - 1}{\cos\,x} = \frac{2\cos\,x\,\cos\,x}{\cos\,x} - \frac{1}{\cos\,x} = 2\cos\,x - \sec\,x$$

Inverse Trigonometric functions

In trigonometry, we give you the value of an angle in degrees or radians and you evaluate its trigonometric value.

In inverse trigonometry, we give the trigonometric value and you evaluate only one value for the angle in radians only (if possible in multiples of π).

The following shows the graph, domain, range and some examples.

Notation: $\sin(\theta) = t \longrightarrow \theta = \text{Sin}^{-1}(t) = \arcsin(t)$

$\cos(\theta) = t \longrightarrow \theta = \text{Cos}^{-1}(t) = \arccos(t)$

$\tan(\theta) = t \longrightarrow \theta = \text{Tan}^{-1}(t) = \arctan(t)$

The inverse sine function

$y = \mathrm{Sin}^{-1}(x) = \arcsin(x)$

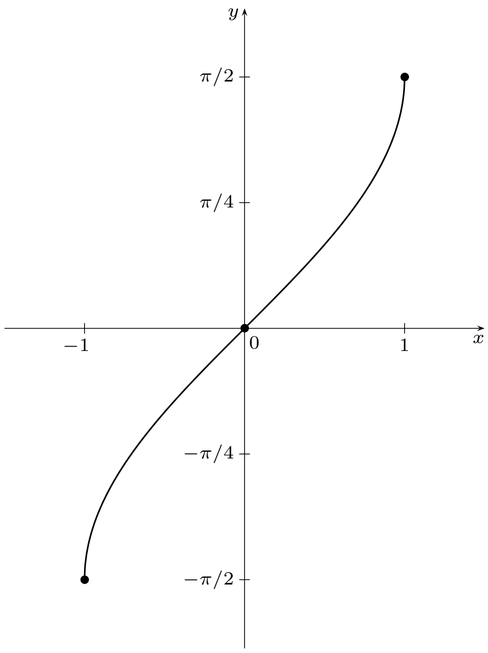
Domain: $[-1, 1]$
(first quadrant from $0 < x < 1$ and fourth quadrant from $-1 < x < 0$)

Range: $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

no asymptotes

$x-$intercept: 0

$y-$intercept: 0



Examples: (1) Evaluate $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$ (2) Evaluate $\arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

(3) Evaluate $\mathrm{Sin}^{-1}(-1) = -\frac{\pi}{2}$

The inverse cosine function

$y = \mathrm{Cos}^{-1}(x) = \arccos(x)$

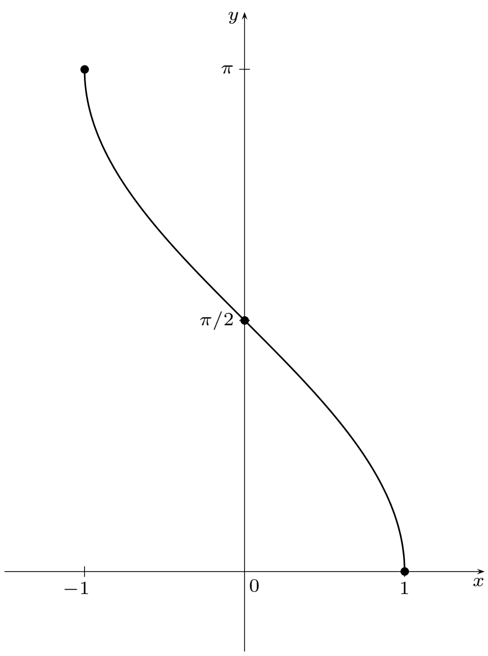
Domain: $[-1, 1]$
(first quadrant from $0 < x < 1$ and second quadrant from $-1 < x < 0$)

Range: $[0, \pi]$

no asymptotes

$x-$intercept: 1

$y-$intercept: $\frac{\pi}{2}$



Examples: (1) Evaluate $\arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$ (2) Evaluate $\arccos\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$

(3) Evaluate $\mathrm{Cos}^{-1}(0) = \frac{\pi}{2}$

The inverse tangent function

$y = \text{Tan}^{-1}(x) = \arctan(x)$

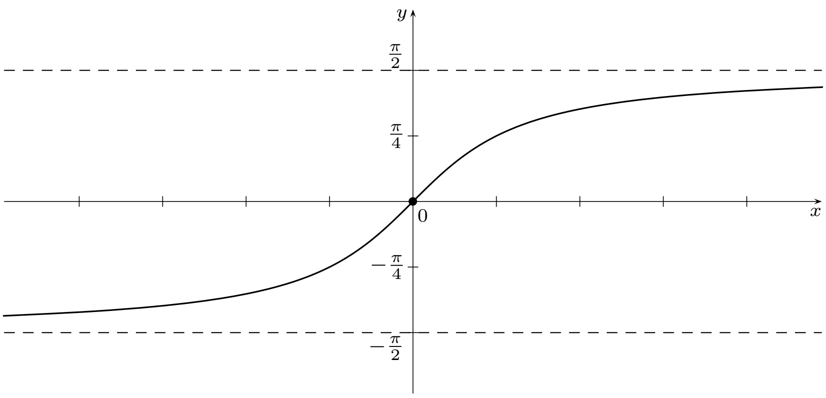
Domain: $\Re$
(first quadrant from $0 < x < +\infty$ and fourth quadrant from $-\infty < x < 0$)

Range: $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

horizontal asymptotes: $y = \pm\frac{\pi}{2}$

$x-$intercept: 0

$y-$intercept: 0



Examples: (1) Evaluate $\arctan\left(-\sqrt{3}\right) = -\frac{\pi}{3}$ (2) Evaluate $\arctan\left(+\infty\right) = \frac{pi}{2}$

(3) Evaluate $\text{Tan}^{-1}\left(1\right) = \frac{\pi}{4}$