Trigonometric Limits

It has been shown in class that:

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{x}{\sin x} = 1$$

$$\lim_{x\to 0} \frac{\cos x - 1}{x} = \frac{1-\cos x}{x} = 0$$

$$\lim_{x\to 0^-} \frac{x}{\cos x - 1} = \lim_{x\to 0^+} \frac{x}{1-\cos x} = +\infty$$

To evaluate trigonometric limits, use these results and the properties of limits shown in another tutorial.

Examples:

(1) $\lim_{x\to \pi} (\cos 2x - \sec x) = \cos(2\pi) - \sec(\pi) = 1-(-1) = 2$

(2) $\lim_{x\to 0} \frac{\sin x}{2x} = \frac{1}{2} \lim_{x\to 0} \frac{\sin x}{x} = \frac{1}{2}(1) = \frac{1}{2}$

(3) $\lim_{x\to 0} \frac{\sin 3x}{x} = 3 \lim_{x\to 0} \frac{\sin x}{x} = 3\,(1) = 3$

(4) Evaluate: $\lim_{x\to 0} \frac{1-\cos^2 x}{x} = \lim_{x\to 0} \frac{(1-\cos x)(1+\cos x)}{x} = \lim_{x\to 0} (1+\cos x) \lim_{x\to 0} \frac{(1-\cos x)}{x} = (1+1)\,(0) = 0$

(5) Evaluate: $= \lim_{x\to 0} \frac{x-1}{\cos 2x} = \frac{(0)-1}{\cos 2(0)} = \frac{-1}{1} = -1$

(6) Evaluate: $\lim_{x\to 0} \frac{\sin^2 x}{x} = \lim_{x\to 0} \frac{\sin x \ \sin x}{x} = \lim_{x\to 0} \sin x \lim_{x\to 0} \frac{\sin x}{x} = (0)\,.\,(1) = 0$

Trigonometric Derivatives formulas

it has been shown in class that:

$$\sin(x) \longrightarrow \cos(x) \quad ; \quad \cos(x) \longrightarrow -\sin(x) \quad ; \quad \tan(x) \longrightarrow \sec^2(x)$$

$$\csc(x) \longrightarrow -\csc(x)\cot(x) \quad ; \quad \sec(x) \longrightarrow \sec(x)\tan(x) \quad ; \quad \cot(x) \longrightarrow -\csc^2(x)$$

If u is a function of x, then the derivatives are:

$$\sin(u) \longrightarrow \cos(u)\,.\,u' \quad ; \quad \cos(u) \longrightarrow -\sin(u)\,.\,u' \quad ; \quad \tan(u) \longrightarrow \sec^2(u)\,.\,u'$$

$$\csc(u) \longrightarrow -\csc(u)\cot(u)\,.\,u' \quad ; \quad \sec(u) \longrightarrow \sec(u)\tan(u)\,.\,u' \quad ; \quad \cot(u) \longrightarrow -\csc^2(u)\,.\,u'$$

Trigonometric Derivative (basic)

(a) Find the derivative of $y = 3\cos x - 5\csc x$

$$y' = \frac{dy}{dx} = -3\sin x - 5\,(-\csc x\ \cot x) = -3\sin x + 5\csc x\cot x$$

(b) Evaluate $y'\left(\frac{\pi}{2}\right)$

$$y'\left(\frac{\pi}{2}\right) = -3\sin\left(\frac{\pi}{2}\right) + 5\csc\left(\frac{\pi}{2}\right)\cot\left(\frac{\pi}{2}\right) = -3(1) + 5(1)(0) = -3$$

Trigonometric Derivative (product)

(a) Find the derivative of $y = \sec x\ (1 + \sin x)$

Use product rule: $(u\,.\,v)' = u'\,.\,v + v'\,.\,u$

$u = \sec x \rightarrow u' = \sec x\tan x$; $v = 1 + \sin x \rightarrow v' = \cos x$

$$y' = \frac{dy}{dx} = \sec x\tan x\ (1 + \sin x) + \cos x\sec x = \sec x\tan x + \sec x\tan x\sin x + 1$$

$$y' = \sec x\tan x + \tan^2 x + 1 = \sec x\tan x + \sec^2 x = \sec x\ (\tan x + \sec x)$$

(b) Evaluate $y'(\pi)$

$$y'(\pi) = \sec(\pi)\Big(\tan(\pi) + \sec(\pi)\Big) = (-1)(0-1) = 1$$

Trigonometric Derivative (quotient)

(a) Find the derivative of $y = \dfrac{\cos x}{\sin x + 1}$

Use quotient rule: $\left(\dfrac{u}{v}\right)' = \dfrac{u'\,.\,v - v'\,.\,u}{v^2}$

$u = \cos x \rightarrow u' = -\sin x$; $v = \sin x + 1 \rightarrow v' = \cos x$

$$y' = \frac{dy}{dx} = \frac{-\sin x\ (\sin x + 1) - \cos x\cos x}{(\sin x + 1)^2} = \frac{-\sin x\ (\sin x + 1) - \cos x\cos x}{(\sin x + 1)^2}$$

$$y' = \frac{-\sin^2 x - \sin x - \cos^2 x}{(\sin x + 1)^2} = \frac{-\left(\sin^2 x + \cos^2 x\right) - \sin x}{(\sin x + 1)^2} = \frac{-1 - \sin x}{(\sin x + 1)^2}$$

$$y' = \frac{-1\,(\sin x + 1)}{(\sin x + 1)^2} = \frac{-1}{\sin x + 1}$$

(b) Evaluate $y'(0) \longrightarrow y'(0) = \dfrac{-1}{\sin(0) + 1} = -1$

Trigonometric Derivative (composition)

(a) Find the derivative of $y = \tan(\sin x - x)$

Use chain rule: $\Big(u(v)\Big)' = u'(v)\,.\,v'$

$u(v) = \tan(\sin x - x) \rightarrow u'(v) = \sec^2(\sin x - x)$; $v = \sin x - x \rightarrow v' = \cos x - 1$

$$y' = \frac{dy}{dx} = \sec^2(\sin x - x)\ .\ (\cos x - 1)$$

(b) Evaluate $y'(0) \longrightarrow y'(0) = \sec^2(\sin x - x)\ .\ (\cos x - 1) = \sec^2(0)\,.\,(1-1) = (1)\,.\,(0) = 0$

Trigonometric Derivative (implicit)

(a) Find y' if $x \sin y = 3y + y \cos x$

use implicit differentiation:

$x \sin y \rightarrow$ product rule: $(1) \sin y + x\ (y' \sin y)$

$3y \rightarrow 3y'$

$y \cos x \rightarrow$ product rule: $(y') \cos x + y\ (-\sin x)$

Derivative equation:

$\sin y + xy' \sin y = 3y' + y' \cos x - y \sin x$

$xy' \sin y - 3y' - y' \cos x = -y \sin x - \sin y$

$-y'(-x \sin y + 3 + \cos x) = -(y \sin x + \sin y)$

$$y' = \frac{y \sin x + \sin y}{3 + \cos x - x \sin y}$$

(b) Evaluate $y'\left(\pi, \frac{\pi}{2}\right) \longrightarrow y'\left(\pi, \frac{\pi}{2}\right) = \frac{\left(\frac{\pi}{2}\right) \sin(\pi) + \sin\left(\frac{\pi}{2}\right)}{3 + \cos(\pi) - (\pi) \sin\left(\frac{\pi}{2}\right)} = \frac{0+1}{3-1-0} = \frac{1}{2}$

Trigonometric Derivative (product) with exponential

Use the exponential derivatives shown in another tutorial.

(a) Find the derivative of $y = \sin(3x)\, e^{2x}$

Use product rule: $(u\,.\,v)' = u'\,.\,v + v'\,.\,u$

$u = \sin(3x) \to u' = 3\,\cos(3x)$; $v = e^{2x} \to v' = 2\,e^{2x}$

$$y' = \frac{dy}{dx} = 3\,\cos(3x)\,e^{2x} + 2\,e^{2x}\,\sin(3x) = e^{2x}\left(3\,\cos(3x) + 2\,\sin(3x)\right)$$

(b) Evaluate $y'(0)$

$$y'(0) = e^{0}\left(3\,\cos(0) + 2\,\sin(0)\right) = 3$$

Trigonometric Derivative (quotient) with exponential

(a) Find the derivative of $y = \dfrac{\cos\,2x}{e^{3x}}$

Use quotient rule: $\left(\dfrac{u}{v}\right)' = \dfrac{u'\,.\,v - v'\,.\,u}{v^2}$

$u = \cos\,2x \to u' = -2\,\sin\,2x$; $v = e^{3x} \to v' = 3\,e^{3x}$

$$y' = \frac{dy}{dx} = \frac{-2\,\sin\,2x\,e^{3x} - 3\,e^{3x}\,\cos\,2x}{\left(e^{3x}\right)^2} = \frac{-\,e^{3x}\,(2\,\sin\,2x + 3\,\cos\,2x)}{\left(e^{3x}\right)^2} = \frac{-\,(2\,\sin\,2x + 3\,\cos\,2x)}{e^{3x}}$$

(b) Evaluate $y'\,(0)$

$$y'(0) = \frac{-\,(2\,\sin\,(0) + 3\,\cos\,(0))}{e^{0}} = \frac{-(0+3)}{1} = -3$$

Trigonometric Derivative (composition) with exponential

(a) Find the derivative of $y = e^{\,2\,\cos\,x - 1}$

Use chain rule: $\left(u(v)\right)' = u'(v)\,.\,v'$

$u(v) = e^{\,2\,\cos\,x-1} \to u'(v) = e^{\,2\,\cos\,x-1}$; $v = 2\,\cos\,x - 1 \to v' = -2\,\sin\,x$

$$y' = \frac{dy}{dx} = e^{\,2\,\cos\,x-1}\,.\,(-2\,\sin\,x) = -2\,\sin\,x\,e^{\,2\,\cos\,x-1}$$

(b) Evaluate $y'\left(\frac{\pi}{3}\right) \longrightarrow y'\left(\frac{\pi}{3}\right) = -2\,\sin\,\left(\frac{\pi}{3}\right)\,e^{\,2\,\cos\left(\frac{\pi}{3}\right)-1} = -2\,\frac{\sqrt{3}}{2}\,e^{0} = -\sqrt{3}$

Trigonometric Derivative (implicit) with exponential

(a) Find y' if $2\,e^y\cos x - \tan y = \cos y + e^x$

use implicit differentiation:

$2\,e^y\cos x \rightarrow$ product rule: $2\,(y'\,e^y)\cos x + 2\,e^y\,(-\sin x)$

$-\tan y \rightarrow -y'\sec^2 y$

$\cos y + e^x \rightarrow$ product rule: $-y'\sin y + e^x$

Derivative equation:

$2\,(y'\,e^y)\cos x + 2\,e^y\,(-\sin x) - y'\sec^2 y = -y'\sin y + e^x$

$2y'\,e^y\cos x - y'\sec^2 y + y'\sin y = 2\,e^y\sin x + e^x$

$y'\left(2\,e^y\cos x - \sec^2 y + \sin y\right) = 2\,e^y\sin x + e^x$

$$y' = \frac{2\,e^y\sin x + e^x}{2\,e^y\cos x - \sec^2 y + \sin y}$$

(b) Evaluate $y'(0,0) \longrightarrow y'(0,0) = \dfrac{2\,e^0\sin(0) + e^0}{2\,e^0\cos(0) - \sec^2(0) + \sin(0)} = \dfrac{0+1}{2-1+0} = 1$

Trigonometric Derivative (product) with logarithm

Use the logarithm derivatives shown in another tutorial.

(a) Find the derivative of $y = \cos(2x)\,\ln(1-x)$

Use product rule: $(u\,.\,v)' = u'\,.\,v + v'\,.\,u$

$u = \cos(2x) \rightarrow u' = -2\sin(2x)$; $v = \ln(1-x) \rightarrow v' = \dfrac{-1}{1-x} = \dfrac{1}{x-1}$

$$y' = \frac{dy}{dx} = -2\sin(2x)\,\ln(1-x) + \frac{1}{x-1}\cos(2x)$$

(b) Evaluate $y'(0)$

$$y'(0) = -2\sin(0)\,\ln(1) + \frac{1}{-1}\cos(0) = -1$$

Trigonometric Derivative (quotient) with logarithm

(a) Find the derivative of $y = \dfrac{\ln(x+1)}{2+\sin\, x}$

Use quotient rule: $\left(\dfrac{u}{v}\right)' = \dfrac{u' \, . \, v - v' \, . \, u}{v^2}$

$u = \ln(x+1) \to u' = \dfrac{1}{x+1}$; $v = 2 + \sin\, x \to v' = \cos\, x$

$$y' = \frac{dy}{dx} = \frac{\frac{1}{x+1}\,(2+\sin\, x) - \cos\, x\, \ln(x+1)}{(2+\sin\, x)^2}$$

(b) Evaluate $y'\,(0) \longrightarrow y'(0) = \dfrac{(1)\left(2+\sin\,(0)\right) - \cos\,(0)\, \ln(1)}{\left(2+\sin\,(0)\right)^2} = \dfrac{1}{2}$

Trigonometric Derivative (composition) with logarithm

(a) Find the derivative of $y = \ln\,(4\,\cos\, x - 1)$

Use chain rule: $\Big(u(v)\Big)' = u'(v) \, . \, v'$

$u(v) = \ln\,(4\,\cos\, x - 1) \to u'(v) = \dfrac{1}{4\,\cos\, x - 1}$; $v = 4\,\cos\, x - 1 \to v' = -4\,\sin\, x$

$$y' = \frac{dy}{dx} = \frac{1}{4\,\cos\, x - 1} \, . \, (-4\,\sin\, x) = \frac{-4\,\sin\, x}{4\,\cos\, x - 1}$$

(b) Evaluate $y'\left(\frac{\pi}{3}\right) \longrightarrow y'\left(\frac{\pi}{3}\right) = \dfrac{-4\,\sin\,\left(\frac{\pi}{3}\right)}{4\,\cos\,\left(\frac{\pi}{3}\right) - 1} = \dfrac{-4\,\left(\frac{\sqrt{3}}{2}\right)}{4\,\left(\frac{1}{2}\right) - 1} = -2\sqrt{3}$

Trigonometric Derivative (implicit) with logarithm

(a) Find y' if $2\sin x + \ln(\sin y) = \ln(\cos x + 1) + \cos y + 2$

use implicit differentiation:

$$2\sin x + \ln(\sin y) \rightarrow\ 2\cos x + \frac{\cos y}{\sin y}\,y' = 2\cos x + y'\cot y$$

$$\ln(\cos x + 1) + \cos y + 2 \rightarrow\ \frac{-\sin x}{\cos x + 1} - y'\sin y$$

Derivative equation:

$$2\cos x + y'\cot y = \frac{-\sin x}{\cos x + 1} - y'\sin y$$

$$y'\cot y + y'\sin y = \frac{-\sin x}{\cos x + 1} - 2\cos x = \frac{-\sin x - 2\cos x\,(\cos x + 1)}{\cos x + 1}$$

$$y'\,(\cot y + \sin y) = \frac{-\sin x - 2\cos x\,(\cos x + 1)}{\cos x + 1}$$

$$y' = \frac{-\sin x - 2\cos x\,(\cos x + 1)}{(\cos x + 1)\,(\cot y + \sin y)}$$

(b) Evaluate $y'\left(\frac{\pi}{2}, \frac{\pi}{2}\right) \longrightarrow\ y'\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = \dfrac{-\sin\left(\frac{\pi}{2}\right) - 2\cos\left(\frac{\pi}{2}\right)\left(\cos\left(\frac{\pi}{2}\right) + 1\right)}{\left(\cos\left(\frac{\pi}{2}\right) + 1\right)\left(\cot\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right)\right)} = \dfrac{-1-0}{(1)(1)} = -1$

Trigonometric Derivative with exponential & logarithm (1)

(a) Find the derivative of $y = e^{\sin(\pi x)} + \cos(\ln x)$

$$y' = \frac{dy}{dx} = \pi\cos(\pi x)\,e^{\sin(\pi x)} + \Big(-\sin(\ln x)\Big)\,\frac{1}{x} = \pi\cos(\pi x)\,e^{\sin(\pi x)} - \frac{\sin(\ln x)}{x}$$

(b) Evaluate $y'(1)$

$$y'(1) = \pi\cos(\pi)\,e^{\sin(\pi)} - \frac{\sin(\ln 1)}{1} = \pi\,(-1)\,e^0 - \sin(0) = -\pi$$

Trigonometric Derivative with exponential & logarithm (2)

(a) Find the derivative of $y = \ln(e^x + \cos x - 1)$

Use chain rule: $\Big(u(v)\Big)' = u'(v)\,.\,v'$

$u(v) = \ln(e^x + \cos x - 1) \to u'(v) = \dfrac{1}{e^x + \cos x - 1}$; $v = e^x + \cos x - 1 \to v' = e^x - \sin x$

$$y' = \frac{dy}{dx} = \frac{1}{e^x + \cos x - 1}\,.\,(e^x - \sin x) = \frac{e^x - \sin x}{e^x + \cos x - 1}$$

(b) Evaluate $y'(0) \longrightarrow y'(0) = \dfrac{e^0 - \sin(0)}{e^0 + \cos(0) - 1} = 1$

Trigonometric Derivative with logarithmic differentiation

(a) Find the derivative of $y = (2x+1)^{\cos x}$

Use logarithmic differentiation: Take ln on both sides

$\ln(y) = \ln(2x+1)^{\cos x}$; use log properties: $\ln(y) = \cos x\ \ln(2x+1)$

implicit differentiation on Left-Hand-Side and product rule on Right-Hand-Side:

$$y'\,\frac{1}{y} = -\sin x\ \ln(2x+1) + \frac{2}{2x+1}\ \cos x$$

$$y' = y\left[-\sin x\ \ln(2x+1) + \frac{2\cos x}{2x+1}\right] = (2x+1)^{\cos x}\left[-\sin x\ \ln(2x+1) + \frac{2\cos x}{2x+1}\right]$$

(b) Evaluate $y'(0)$

$$y'(0) = (1)^{\cos(0)}\left[-\sin(0)\ \ln(1) + \frac{2\cos(0)}{1}\right] = 2$$