What is the Antiderivative?

In a derivative problem, a function $f(x)$ is given and you find the derivative $f'(x)$ using the formulas and rules of derivatives shown in a previous tutorial.

In an antiderivative problem, the derivative $f'(x)$ is given and you find a function $f(x)$ using the formulas of antiderivatives shown later in this tutorial.

In other words, the antiderivative is the reverse question to the derivative question.

Power Rule for antiderivatives

Some of the examples shown in the tutorial "Rules of Derivative", are used to examine the following patterns:
The term has form of $k\,x^n$ where k is the coefficient and n is the exponent ; k and n are real numbers , x is the variable

The antiderivative of x^2 is x^2+(a constant), namely:
$x^2+4\,;\ x^2-10\,;\ x^2+C$ where C is called constant of integration.
more examples: $6x^2$ has antiderivative $2x^3+C$
$15x^4$ has antiderivative $3x^5+C$
$3=3x^0$ has antiderivative $3x+C$

The Power Rule for antiderivatives: $\boxed{kx^n \implies k\,\dfrac{x^{n+1}}{n+1}+C\ ;\ n\neq -1}$ This formula does not work for $n=-1$

Examples: Find the antiderivative of each term:

1. $4\sqrt{x}=4x^{1/2} \implies 4\,\dfrac{x^{3/2}}{3/2}+C=\frac{8}{3}\,x^{3/2}+C$

2. $16\,\sqrt[3]{x}=16x^{1/3} \implies 16\,\dfrac{x^{4/3}}{4/3}+C=12x^{4/3}+C$

3. $\dfrac{5}{x^3}=5x^{-3} \implies 5\,\dfrac{x^{-2}}{-2}+C=-\frac{5}{2}\,x^{-2}+C=-\dfrac{5}{2x^2}+C$

4. $\dfrac{10}{\sqrt{x}}=10x^{-1/2} \implies 10\dfrac{x^{1/2}}{1/2}+C=20x^{1/2}+C=20\,\sqrt{x}+C$

5. $\dfrac{6}{\sqrt[3]{x}}=6x^{-1/3} \implies 6\,\dfrac{x^{2/3}}{2/3}+C=9x^{2/3}+C$

Other basic antiderivatives formulas:

$\boxed{\text{constant } k=kx^0 \implies kx+C}$

$\boxed{n=-1 \longrightarrow x^{-1}=\dfrac{1}{x} \implies \ln|x|+C}$

$\boxed{e^x \implies e^x+C}$

$\boxed{a^x \implies \dfrac{a^x}{\ln(a)}+C}$

These formulas are verified if the derivative is applied to each antiderivative answer.

Notation and Symbols of Antiderivatives:

Find the antiderivative of $10x^7$; instead of this question in words, we use the following: $\int (10x^7)\, dx$

where $\int$ is called the integration sign (like S letter with long neck)

$(10x^7)$ the term you find the antiderivative in parenthesis

dx is called the differential of x

dx indicates that the answer is in terms of x

dx does not get involved in the antiderivative (also called indefinite integral)

<u>Solution</u>: $\int (10x^7)\, dx = 10\,\frac{x^8}{8} + C = \frac{5}{4}\,x^8 + C$

Sum and Difference of terms:

To find the antiderivative of a sum or a difference of terms, use appropriate basic formulas for each term.

Examples :

$$\int \left(3x^5 - \frac{5}{\sqrt{x}}\right)\, dx = \int \left(3x^5 - 5x^{-1/2}\right)\, dx = 3\,\frac{x^6}{6} - 5\,\frac{x^{1/2}}{1/2} + C = \frac{1}{2}\,x^6 - 10x^{1/2} + C$$

$$\int \left(12x^{1/3} + 8x^3\right)\, dx = 12\,\frac{x^{4/3}}{4/3} + 8\,\frac{x^4}{4} + C = 9x^{4/3} + 2x^4 + C$$

$$\int \left(\frac{2}{x} - 4\,e^x\right)\, dx = \int \left(2x^{-1} - 4\,e^x\right)\, dx = 2\,\ln|x| - 4\,e^x + C$$

$$\int \left(2 + \frac{3}{v^2}\right)\, dv = \int \left(2v^0 + 3v^{-2}\right)\, dv = 2v + 3\,\frac{v^{-1}}{-1} + C = 2v - \frac{3}{v} + C$$

$$\int \left(x - 3 + \sqrt{x}\,\right)\, dx = \int \left(x - 3x^0 + x^{1/2}\right)\, dx = \frac{x^2}{2} - 3x + \frac{x^{3/2}}{3/2} + C = \frac{1}{2}\,x^2 - 3x + \frac{2}{3}\,x^{3/2} + C$$

Product of terms:

To find the antiderivative of a product of terms, must first multiply, reduce if necessary, then use appropriate basic formulas for each term.

Examples :

$$\int x\,(1 - x)\, dx = \int \left(x - x^2\right)\, dx = \frac{x^2}{2} - \frac{x^3}{3} + C = \frac{1}{2}\,x^2 - \frac{1}{3}\,x^3 + C$$

$$\int x^2\left(4 + x^3\right)\, dx = \int \left(4x^2 + x^5\right)\, dx = 4\,\frac{x^3}{3} + \frac{x^6}{6} + C = \frac{4}{3}\,x^3 + \frac{1}{6}\,x^6 + C$$

$$\int x^{-1}\left(\sqrt{x} + 3\right)\, dx = \int \left(x^{-1/2} + 3\,x^{-1}\right)\, dx = \frac{x^{1/2}}{1/2} + 3\,\ln|x| + C = 2\,x^{1/2} + 3\,\ln|x| + C$$

Product of terms:

To find the antiderivative of a product of terms, must first multiply, reduce if necessary, then use appropriate basic formulas for each term.

Examples :

$$\int (x+1)(x-2)\,dx = \int \left(x^2 - x - 2\right)\,dx = \frac{x^3}{3} - \frac{x^2}{2} - 2x + C = \frac{1}{3}\,x^3 - \frac{1}{2}\,x^2 - 2x + C$$

$$\int (x^2+3)^2\,dx = \int \left(x^4 + 6x^2 + 9\right)\,dx = \frac{x^5}{5} + 6\,\frac{x^3}{3} + 9x + C = \frac{1}{5}\,x^5 + 2x^3 + 9x + C$$

Quotient of terms:

To find the antiderivative of a quotient of terms, must first divide, reduce if necessary, then use appropriate basic formulas for each term.

Examples:

$$\int \frac{2x^3 - 6x^5}{x^2}\,dx = \int \left(2x^3 - 6x^5\right)x^{-2}\,dx = \int \left(2x - 6x^3\right)\,dx = 2\,\frac{x^2}{2} - 6\,\frac{x^4}{4} + C = x^2 - \frac{3}{2}\,x^4 + C$$

$$\int \frac{4x^3 - 6x\,e^x}{2x}\,dx = \int \left(\frac{4x^3}{2x} - \frac{6x\,e^x}{2x}\right)\,dx = \int \left(2x^2 - 3\,e^x\right)\,dx = 2\,\frac{x^3}{3} - 3\,e^x + C = \frac{2}{3}\,x^3 - 3\,e^x + C$$

$$\int \frac{3\sqrt{x} - 5x^2}{x^{3/2}}\,dx = \int \left(3x^{1/2} - 5x^2\right)x^{-3/2}\,dx = \int \left(3x^{-1} - 5x^{1/2}\right)\,dx$$
$$= 3\,\ln|x| - 5\,\frac{x^{3/2}}{3/2} + C = 3\,\ln|x| - \tfrac{10}{3}\,x^{3/2} + C$$

$$\int \frac{2x - 5x^2\,e^x}{x^2}\,dx = \int \left(2x - 5x^2\,e^x\right)x^{-2}\,dx = \int \left(2x^{-1} - 5\,e^x\right)\,dx = 2\,\ln|x| - 5\,e^x + C$$

$$\int \frac{4x^2 - 16}{x-2}\,dx = \int \frac{(4x+8)(x-2)}{x-2}\,dx = \int (4x+8)\,dx = 4\,\frac{x^2}{2} + 8x + C = 2x^2 + 8x + C$$

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more examples:

$$\int \left[\,6x(x^2+1) + 3\,\right]dx = \int \left[\,6x^3 + 6x + 3\,\right]dx = 6\,\frac{x^4}{4} + 6\,\frac{x^2}{2} + 3x + C = \frac{3}{2}\,x^4 + 3x^2 + 3x + C$$

$$\int \left[\,3\,(2^x) - 6\,e^x + 10x^{-1}\,\right]dx = 3\,\frac{2^x}{\ln(2)} - 6\,e^x + 10\,\ln|x| + C$$

$$\int \left(14\sqrt{x^5} - 15\sqrt[3]{x^2} + 12\right)dx = \int \left(14x^{5/2} - 15x^{2/3} + 12\right)dx = 14\,\frac{x^{7/2}}{7/2} - 15\,\frac{x^{5/3}}{5/3} + 12x + C = 6x^{7/2} - 9x^{5/3} + 12x + C$$

$$\int \frac{x\,3^x - 4 + 6x\,e^x}{x}\,dx = \int \left(x\,3^x - 4 + 6x\,e^x\right)x^{-1}\,dx = \int \left(3^x - 4x^{-1} + 6\,e^x\right)\,dx = \frac{3^x}{\ln(3)} - 4\,\ln|x| + 6\,e^x + C$$

In next example, see importance of differential dr

$$\int 15\,dr = 15r + C$$

Indefinite Integral with condition:

The problem gives the first derivative of $f(x)$ with a given condition.
To solve for $f(x)$, find the antiderivative and use the condition to solve for C.

<u>Problem 1</u>: Given $f'(x) = 48x$ with condition $f(1) = 15$. Find $f(x)$

<u>Solution 1</u>: $f(x) = \int (48x)\,dx = 24x^2 + C$

replace $x = 1$ and $f(1) = 15$ to solve for C:

$15 = 24(1)^2 + C \implies C = -9$

<u>Answer 1</u>: $f(x) = 24x^2 - 9$

<u>Problem 2</u>: Given $f'(x) = 12 - 6x$ with condition $f(-1) = 8$. Find $f(x)$

<u>Solution 2</u>: $f(x) = \int (12 - 6x)\,dx = 12x - 3x^2 + C$

replace $x = -1$ and $f(-1) = 8$ to solve for C:

$8 = 12(-1) - 3(-1)^2 + C \implies C = 23$

<u>Answer 2</u>: $f(x) = 12x - 3x^2 + 23$

Indefinite Integral with conditions:

The problem gives the second derivative of $f(x)$ with 2 given conditions.
To solve for $f(x)$, find the antiderivative twice and use the conditions to solve 2 constants of integration.

Problem 1: Given $f''(x) = 4\,e^x + 12x$ with conditions $f'(0) = 5\,,\ f(0) = 1\,.$ Find $f(x)$

Solution 1: $f'(x) = \displaystyle\int (4\,e^x + 12x)\,dx = 4\,e^x + 6x^2 + C_1$

replace $x = 0$ and $f'(0) = 5$ to solve for C_1 :

$5 = 4e^0 + 12(0)^2 + C_1 \implies C_1 = 1 \implies f'(x) = 4\,e^x + 6x^2 + 1$

$f(x) = \displaystyle\int (4\,e^x + 6x^2 + 1)\,dx = 4\,e^x + 2x^3 + x + C_2$

replace $x = 0$ and $f(0) = 1$ to solve for C_2 :

$1 = 4\,e^0 + 2(0)^3 + 0 + C_2 \implies C_2 = -3$

Answer 1: $f(x) = 4\,e^x + 2x^3 + x - 3$

Problem 2: Given $f''(x) = \dfrac{6}{x^2} - 4$ with conditions $f'(1) = 2\,,\ f(1) = 9\,.$ Find $f(x)$

Solution 2: $f'(x) = \displaystyle\int (6x^{-2} - 4)\,dx = -6x^{-1} - 4x + C_1$

replace $x = 1$ and $f'(1) = 2$ to solve for C_1 :

$2 = -6(1)^{-1} - 4(1) + C_1 \implies C_1 = 12 \implies f'(x) = -6x^{-1} - 4x + 12$

$f(x) = \displaystyle\int (-6x^{-1} - 4x + 12)\,dx = -6\,\ln|x| - 2x^2 + 12x + C_2$

replace $x = 1$ and $f(1) = 9$ to solve for C_2 :

$9 = -6\,\ln|1| - 2(1)^2 + 12(1) + C_2 \implies C_2 = -1$

Answer 2: $f(x) = -6\,\ln|x| - 2x^2 + 12x - 1$

Problem 3: Given $f''(x) = 15\sqrt{x} - 12$ with conditions $f'(4) = 1\,,\ f(1) = 3\,.$ Find $f(x)$

Solution 3: $f'(x) = \displaystyle\int (15x^{1/2} - 12)\,dx = 10x^{3/2} - 12x + C_1$

replace $x = 4$ and $f'(4) = 1$ to solve for C_1 :

$1 = 10(4)^{3/2} - 12(4) + C_1 \implies C_1 = -30 \implies f'(x) = 10x^{3/2} - 12x - 30$

$f(x) = \displaystyle\int (10x^{3/2} - 12x - 30)\,dx = 4x^{5/2} - 6x^2 - 30x + C_2$

replace $x = 1$ and $f(1) = 3$ to solve for C_2 :

$3 = 4(1)^{5/2} - 6(1)^2 - 30x + C_2 \implies C_2 = 35$

Answer 3: $f(x) = 4x^{5/2} - 6x^2 - 30x + 35$