Area and the Riemann Sum

Let $f(x)$ be a continuous function and

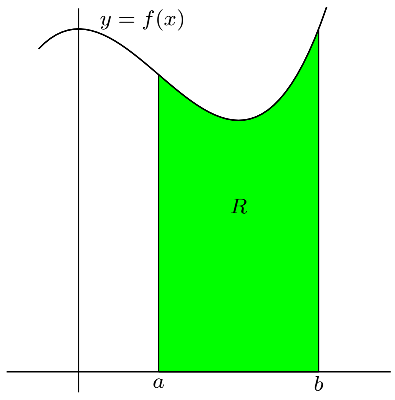
$f(x) > 0$ on a closed interval $[a, b]$ as shown

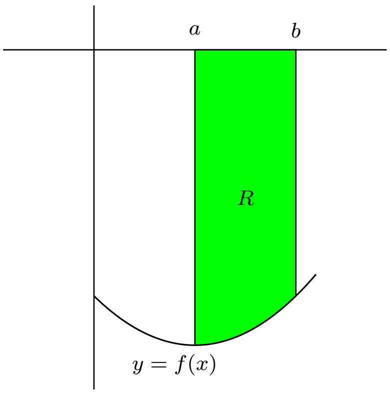
on the graph.

The Riemann Sum theory shows that the area of

the region R has area= $\int_a^b f(x)\,dx$

The area is evaluated by the definite integral.





The Riemann Sum theory also shows that

if $f(x) < 0$ on $[a, b]$,

the definite integral will give a negative answer

(negative area ??)

To avoid this situation, always sketch the region of the area;

if the region is above the $x-$axis, it's fine;

if the region is below the $x-$axis, take the absolute value of the definite integral.

Example 1:

Find the area of the region bounded by $f(x) = 4x^2$, the $x-$axis from $x = 3$ to $x = 6$.

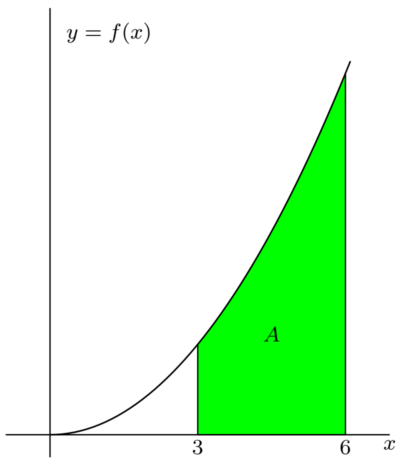
The graph shows that the region is above the $x-$axis.

Then the area: $A = \int_3^6 4x^2\,dx = \left(\frac{4}{3}x^3\right)\Big|_3^6$

$A = \left[\frac{4}{3}(6)^3\right] - \left[\frac{4}{3}(3)^3\right] = 288 - 36 = 252$ square units

Note Area is always in square units.

Definite integrals have no units.



Example 2:

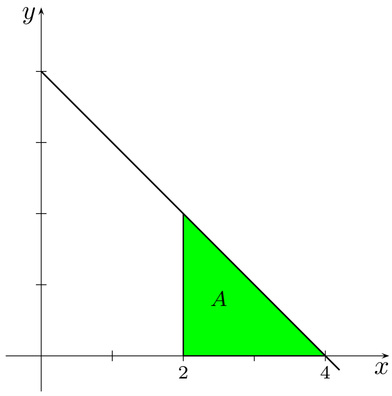
Find the area of the region bounded by $f(x) = 4 - x$, the $x-$axis from $x = 2$ to $x = 4$.

The graph shows that the region is above the $x-$axis.

Then the area: $A = \int_2^4 (4 - x)\,dx = \left(4x - \frac{1}{2}\,x^2\right)\Big|_2^4$

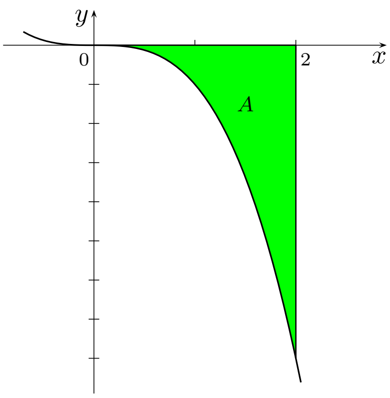
$A = \left[4(4) - \frac{1}{2}\,(4)^2\right] - \left[4(2) - \frac{1}{2}\,(2)^2\right]$

$A = 8 - 6 = 2$ square units



Example 3:

Find the area of the region bounded by $f(x) = -x^3$, the $x-$axis from $x = 0$ to $x = 2$.



The graph shows that the region is below the $x-$axis;
must insert a negative sign in the definite integral.

Then the area: $A = \int_0^2 -(-x^3)\,dx = \left(\frac{1}{4}\,x^4\right)\Big|_0^2$

$A = \left[\frac{1}{4}\,(2)^4\right] - \left[\frac{1}{4}\,(0)^4\right]$

$A = 4 - 0 = 4$ square units

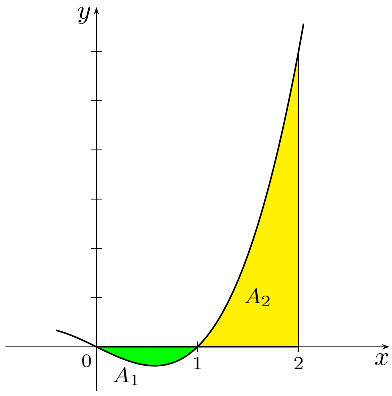
Note: the answer of the area is always a positive number

Example 4:

Find the area of the region bounded by $f(x) = x^3 - x$, the $x-$axis from $x = 0$ to $x = 2$.

The graph shows that the region is in 2 parts.

region A_1 from $x = 0$ to $x = 1$,
the region is below the $x-$axis;
must insert a negative sign in the definite integral
region A_2 from $x = 1$ to $x = 2$,
the region is above the $x-$axis;



Therefore:

$$A_1 = \int_0^1 -(x^3 - x)\,dx = \left(-\frac{1}{4}x^4 + \frac{1}{2}x^2\right)\Bigg|_0^1$$

$$A_1 = \left[-\frac{1}{4}(1)^4 + \frac{1}{2}(1)^2\right] - \left[-\frac{1}{4}(0)^4 + \frac{1}{2}(0)^2\right] = \frac{1}{4}$$

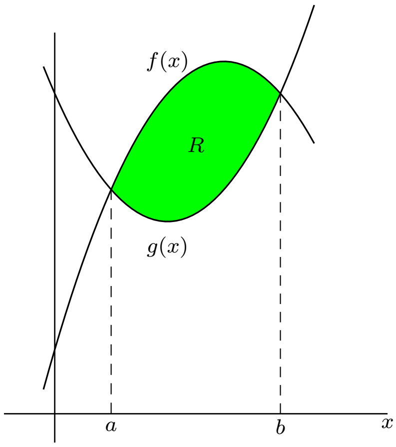
$$A_2 = \int_1^2 (x^3 - x)\,dx = \left(\frac{1}{4}x^4 - \frac{1}{2}x^2\right)\Bigg|_1^2$$

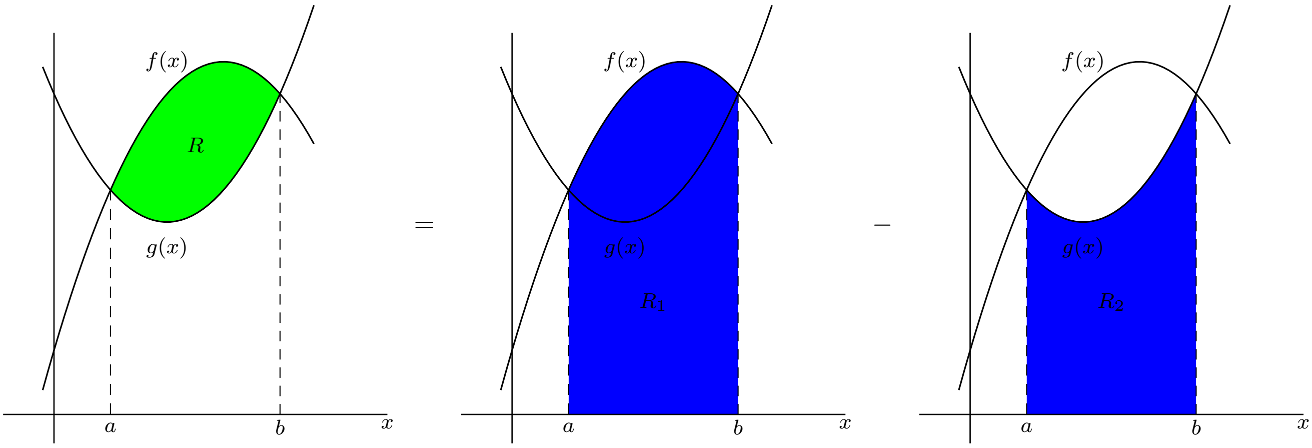
$$A_2 = \left[\frac{1}{4}(2)^4 - \frac{1}{2}(2)^2\right] - \left[\frac{1}{4}(1)^4 - \frac{1}{2}(1)^2\right] = \frac{9}{4}$$

total area: $A_1 + A_2 = \frac{1}{4} + \frac{9}{4} = \frac{5}{2} = 2.5$ square units

Area of a region between two graphs

Let two continuous functions $f(x)$ and $g(x)$
To find the points of intersection: make $f(x) = g(x)$ and solve for x.
Let them be $x = a$ and $x = b$,
then the points of intersection are:
$\Big(a, f(a) = g(a)\Big)$ and $\Big(b, f(b) = g(b)\Big)$.
Find the area of the region between the two graphs (see graph)
Area= $\displaystyle\int_a^b [f(x) - g(x)]\, dx$
Visual proof is shown below.
In general, the region R is evaluated between $x = a$ and $x = b$.
Integrate the difference between highest y function and lowest y function.
Note: Remember to graph the region between 2 curves if requested.





$$\text{Area} = \int_a^b [f(x) - g(x)]\, dx = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$$

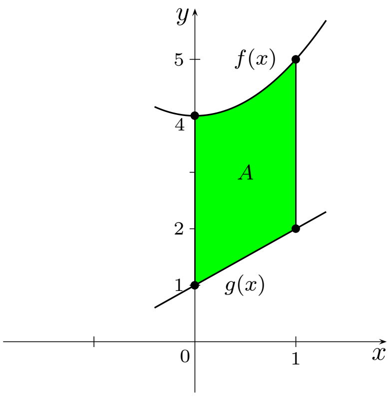
$$\text{Area of } R = \int_a^b [f(x) - g(x)]\, dx = \text{Area of } R_1 - \text{Area of } R_2$$

$$\boxed{\text{Area of } R = \int_a^b [f(x) - g(x)]\, dx}$$

Note: Remember to graph the region between 2 curves if requested.

Example 5:

Find the area of the region bounded by $f(x) = x^2 + 4$, $g(x) = x + 1$ from $x = 0$ to $x = 1$.



The graph shows no point of intersection since

$x^2 + 4 = x + 1 \to x^2 - x + 3 = 0 \to$ no solution.

Area of the region: $A = \int_0^1 \left[f(x) - g(x)\right] \, dx$

$$A = \int_0^1 \left[(x^2 + 4) - (x + 1)\right] \, dx = \int_0^1 \left[x^2 - x + 3\right] \, dx$$

$$A == \left(\frac{1}{3}\,x^3 - \frac{1}{2}\,x^2 + 3x\right)\Big|_0^1$$

$$A = \left[\frac{1}{3}\,(1)^3 - \frac{1}{2}\,(1)^2 + 3(1)\right] - \left[\frac{1}{3}\,(0)^3 - \frac{1}{2}\,(0)^2 + 3(0)\right]$$

$$A = \frac{17}{6} - 0 = \frac{17}{6} \approx 2.83 \text{ square units}$$

Example 6:

Find the area of the region bounded by $f(x) = -x^2 + 1$, $g(x) = -x - 1$.

The graph shows two points of intersection:

$-x - 1 = -x^2 + 1 \to x^2 - x - 2 = 0 \to (x - 2)(x + 1) = 0$

$x = -1 \to y = f(-1) = g(-1) = 0$

$x = 2 \to y = f(2) = g(2) = -3$

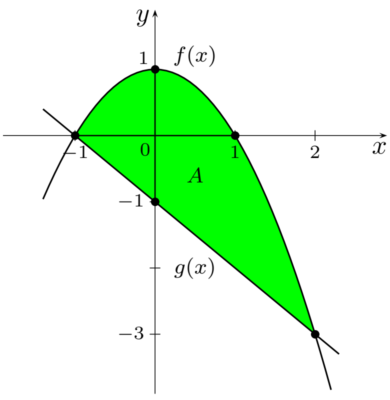
Area of the region: $A = \int_{-1}^2 \left[f(x) - g(x)\right] \, dx$

$$A = \int_{-1}^2 \left[(-x^2 + 1) - (-x - 1)\right] \, dx = \int_{-1}^2 \left[-x^2 + x + 2\right] \, dx$$

$$A == \left(-\frac{1}{3}\,x^3 + \frac{1}{2}\,x^2 + 2x\right)\Big|_{-1}^2$$

$$A = \left[-\frac{1}{3}\,(2)^3 + \frac{1}{2}\,(2)^2 + 2(2)\right] - \left[-\frac{1}{3}\,(-1)^3 + \frac{1}{2}\,(-1)^2 + 2(-1)\right]$$

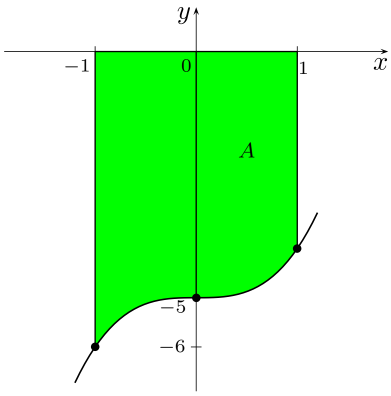
$$A = \frac{10}{3} - \frac{-7}{6} = \frac{9}{2} = 4.5 \text{ square units}$$



Example 7:

Find the area of the region bounded by $f(x) = x^3 - 5$, the $x-$axis from $x = -1$ to $x = 1$.

The graph shows no point of intersection between $x = -1$ and $x = 1$



Area of the region: $A = \int_{-1}^{1} [g(x) - f(x)]\, dx$

where $f(x) = x^3 - 5$ and $g(x) = 0$ ($x-$axis)

$A = \int_{-1}^{1} \left[(0) - (x^3 - 5)\right] dx = \int_{-1}^{1} \left[-x^3 + 5\right] dx$

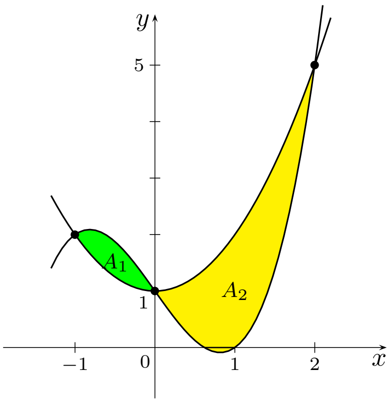
$A = \left(-\frac{1}{4}\, x^4 + 5x\right)\Big|_{-1}^{1}$

$A = \left[-\frac{1}{4}\,(-1)^4 + 5(-1)\right] - \left[-\frac{1}{4}\,(1)^4 + 5(1)\right]$

$A = \frac{19}{4} - \frac{-21}{4} = 10$ square units

Example 8:

Find the area of the region bounded by $f(x) = x^3 - 2x + 1$, $g(x) = x^2 + 1$.



The graph shows that three points of intersection:

$x^3 - 2x + 1 = x^2 + 1 \rightarrow x^3 - x^2 - 2x = 0 \rightarrow x(x-2)(x+1) = 0$

$x = -1 \rightarrow y = f(-1) = g(-1) = 2$

$x = 0 \rightarrow y = f(0) = g(0) = 1$; $x = 2 \rightarrow y = f(2) = g(2) = 5$

Area of the region: $A = A_1 + A_2$

$A = \int_{-1}^{0} [f(x) - g(x)]\, dx + \int_{0}^{2} [g(x) - f(x)]\, dx$

$A_1 = \int_{-1}^{0} \left[(x^3 - 2x + 1) - (x^2 + 1)\right] dx = \int_{-1}^{0} \left[x^3 - x^2 - 2x\right] dx$

$A_1 = \left[\frac{1}{4}\, x^4 - \frac{1}{3}\, x^3 - x^2\right]\Big|_{-1}^{0} = \frac{5}{12}$

$A_2 = \int_{0}^{2} \left[(x^2 + 1) - (x^3 - 2x + 1)\right] dx = \int_{0}^{2} \left[-x^3 + x^2 + 2x\right] dx = \left[-\frac{1}{4}\, x^4 + \frac{1}{3}\, x^3 + x^2\right]\Big|_{0}^{2} = \frac{8}{3}$

total area: $A_1 + A_2 = \frac{5}{12} + \frac{8}{3} = \frac{37}{12} \approx 3.083$ square units

Example 9:

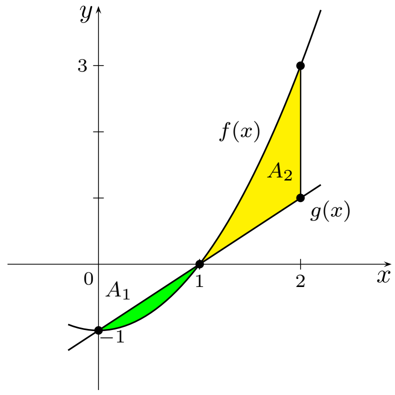
Find the area of the region bounded by $f(x) = x^2 - 1$, $g(x) = x - 1$ from $x = 0$ to $x = 2$.

The graph shows that two points of intersection:

$x^2 - 1 = x - 1 \rightarrow x^2 - x = 0 \rightarrow x(x-1) = 0$

$x = 0 \rightarrow y = f(0) = g(0) = -1$

$x = 1 \rightarrow y = f(1) = g(1) = 0$; $x = 2 \rightarrow f(2) = 3$; $g(2) = 1$



Area of the region: $A = A_1 + A_2$

$$A = \int_0^1 [g(x) - f(x)]\, dx + \int_1^2 [f(x) - g(x)]\, dx$$

$$A_1 = \int_0^1 \left[(x-1) - (x^2 - 1)\right] dx = \int_0^1 \left[-x^2 + x\right] dx$$

$$A_1 = \left[-\frac{1}{3}x^3 + \frac{1}{2}x^2\right]\Bigg|_0^1 = \frac{1}{6}$$

$$A_2 = \int_1^2 \left[(x^2 - 1) - (x - 1)\right] dx = \int_1^2 \left[x^2 - x\right] dx = \left[\frac{1}{3}x^3 - \frac{1}{2}x^2\right]\Bigg|_1^2 = \frac{1}{2}$$

total area: $A_1 + A_2 = \frac{1}{6} + \frac{1}{2} = \frac{2}{3} \approx 0.67$ square unit

Consumer Surplus

Given the demand $D(x) = p(x)$ and
the supply $S(x) = p(x)$

To find the equilibrium point: make $D(x) = S(x)$

and solve for x.

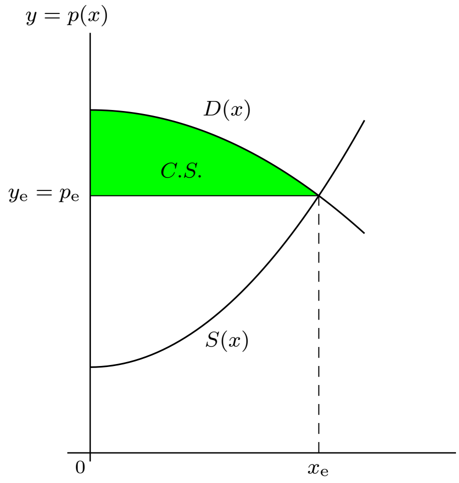
Let x_e and $y_\mathrm{e} = p_\mathrm{e}$ be the coordinates of equilibrium point;

then the consumer surplus is evaluated:

$$\text{C.S.} = \int_0^{x_\mathrm{e}} [\, D(x) - y_\mathrm{e} \,]\, dx$$

C.S. is the region as shown on the graph.

Note: Remember to graph the region between 2 curves if requested.



Producer Surplus

Given the demand $D(x) = p(x)$ and
the supply $S(x) = p(x)$

To find the equilibrium point: make $D(x) = S(x)$

and solve for x.

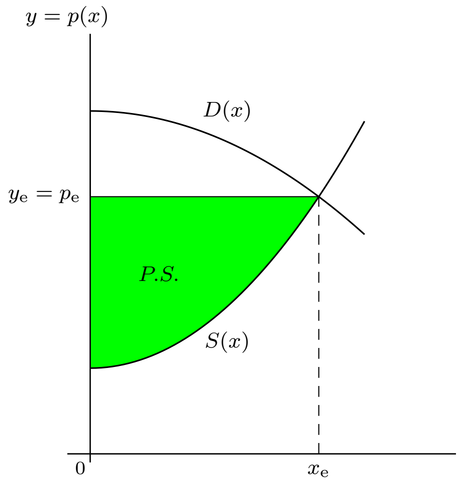
Let x_e and $y_\mathrm{e} = p_\mathrm{e}$ be the coordinates of equilibrium point;

then the producer surplus is evaluated:

$$\text{P.S.} = \int_0^{x_\mathrm{e}} [\, y_\mathrm{e} - S(x) \,]\, dx$$

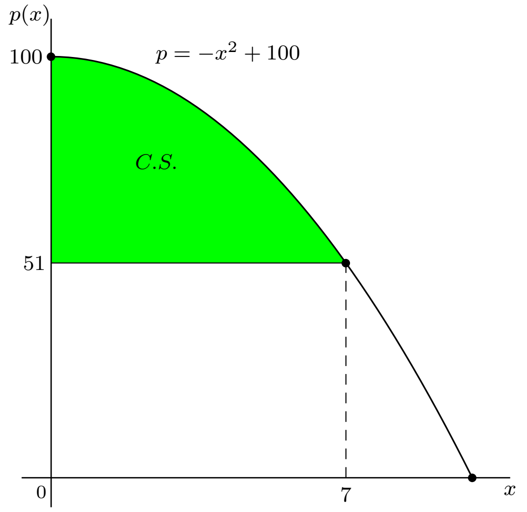
P.S. is the region as shown on the graph.

Note: Remember to graph the region between 2 curves if requested.



Example 10:

Given the demand $p = -x^2 + 100$ at the equilibrium quantity of 7 units, sketch and identify the $C.S.$ region, then evaluate the $C.S.$



The graph shows the $C.S.$ region.

To find the equilibrium price: replace $x_e = 7$

in the demand equation: $p_e = -7^2 + 100 = 51$

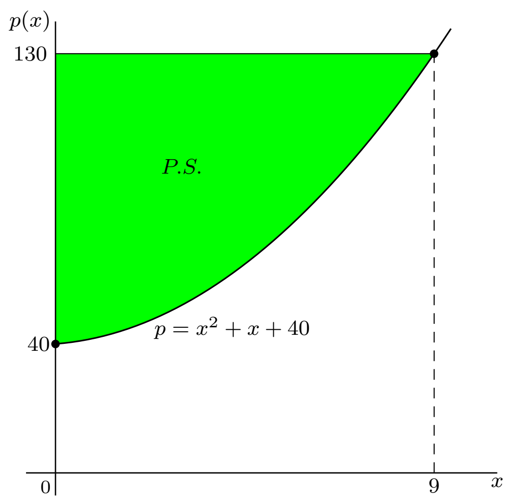
$$C.S. = \int_0^{x_e} [\text{ demand} - p_e \,]\, dx$$

$$C.S. = \int_0^{7} [\,(-x^2 + 100) - 51\,]\, dx = \int_0^{7} [\,-x^2 + 49\,]\, dx$$

$$C.S. = \left(-\frac{1}{3}x^3 + 49x\right)\Bigg|_0^7 = \frac{686}{3} - 0 = \frac{686}{3} \approx \$228.67$$

Example 11:

Given the supply $p = x^2 + x + 40$ at the equilibrium price of 130 \$/per unit, sketch and identify the $P.S.$ region, then evaluate the $P.S.$



The graph shows the $P.S.$ region.

To find the equilibrium quantity: replace $p_e = 130$

in the demand equation: $130 = x^2 + x + 40$

$x^2 + x - 90 = 0 \rightarrow (x + 10)(x - 9) = 0 \rightarrow x_e = 9$

$$P.S. = \int_0^{x_e} [\, p_e - \text{supply}\,]\, dx$$

$$P.S. = \int_0^{9} [\,130 - (x^2 + x + 40)\,]\, dx$$

$$P.S. = \int_0^{9} [\,-x^2 - x + 90\,]\, dx = \left(-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 90x\right)\Bigg|_0^9$$

$$P.S. = \frac{1053}{2} - 0 = \frac{1053}{2} \approx \$526.50$$

Example 12:

Given the demand $p = -x^2 + 64$ and the supply $p = 2x + 29$, sketch and identify the $C.S.$ and the $P.S.$ regions, then evaluate the $P.S.$

To find the equilibrium price: make demand = supply

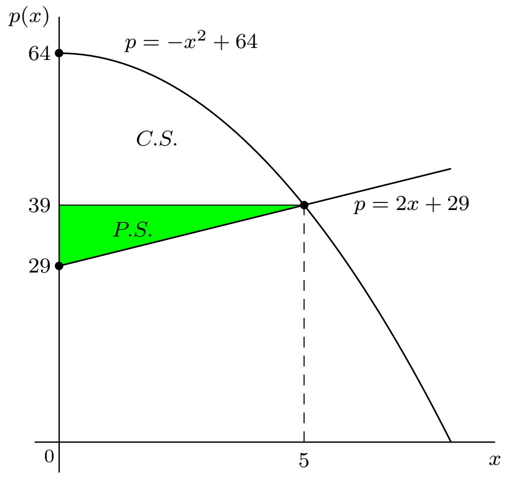
$-x^2 + 64 = 2x + 29 \rightarrow x^2 + 2x - 35 = 0 \rightarrow (x-5)(x+7) = 0$

The equilibrium point at $(5, 39)$

$$P.S. = \int_0^{x_e} [\ p_e - \text{supply}\ ]\ dx$$

$$P.S. = \int_0^{5} [\ 39 - (2x+29)\ ]\ dx = \int_0^{5} [\ -2x+10\ ]\ dx$$

$$P.S. = \left(x^2 + 10x\right)\Big|_0^5 = 25 - 0 = \$25.00$$



Area of a region between two curves (one or two is/are function(s) of $f(y)$)

If one of the two curves is a function of y like $x = f(y)$

To find the points of intersection: make $f(y) = g(y)$ and solve for y.

Let them be $y = a$ and $y = b$,

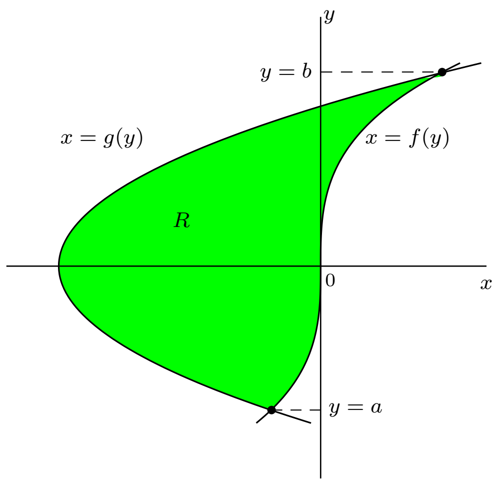
then the points of intersection are:

$\left(f(a) = g(a), a\right)$ and $\left(f(b) = g(b), b\right)$.

Find the area of the region between the two graphs

by integrating with respect to y (see graph)

$$\text{Area} = \int_a^b [f(y) - g(y)]\ dy$$



Visual proof is shown below.

In general, the region R is evaluated between $y = a$ and $y = b$.

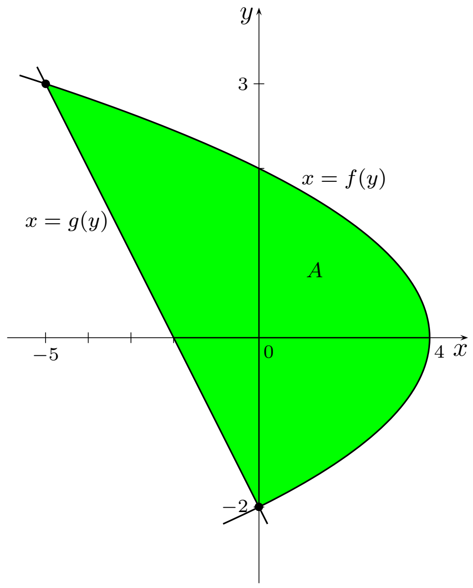
Integrate the difference between right x function and left x function.

Note: Remember to graph the region between 2 curves if requested.

Example 13:

Find the area of the region bounded by $f(y) = -y^2 + 4$ and $g(y) = -y - 2$

The graph shows the shaded region has two points of intersections:



$g(y) = f(y) \implies -y - 2 = -y^2 + 4$

$y^2 - y - 6 = 0 \rightarrow (y-3)(y+2) = 0$

$y = 3 \rightarrow x = f(3) = g(3) = -5$

$y = -2 \rightarrow x = f(-2) = g(-2) = 0$

Area: $A = \displaystyle\int_{-2}^{3} [f(y) - g(y)]\, dy$

$A = \displaystyle\int_{-2}^{3} \left[(-y^2 + 4) - (-y - 2)\right]\, dy$

$A = \displaystyle\int_{-2}^{3} \left[-y^2 + y + 6\right]\, dy$

$A = \left(-\frac{1}{3}y^3 + \frac{1}{2}\,y^2 + 6y\right)\Big|_{-2}^{3}$

$A = \frac{27}{2} - \frac{-22}{3} = \frac{125}{6} \approx 20.83$ square units

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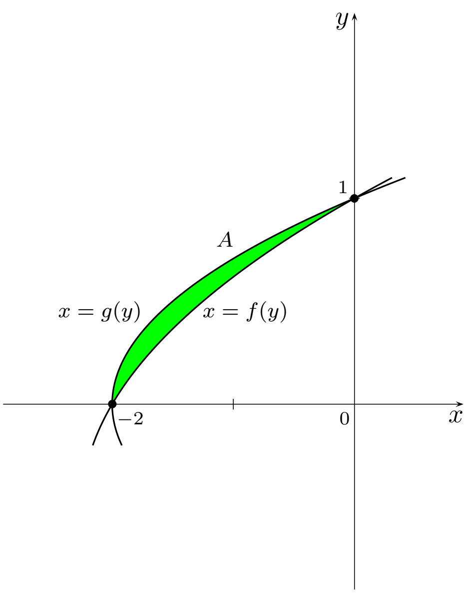
Example 14:

Find the area of the region bounded by $f(y) = y^2 + y - 2$ and $g(y) = 2y^2 - 2$



The graph shows the shaded region has two points of intersections:

$g(y) = f(y) \implies 2y^2 - 2 = y^2 + y - 2$

$y^2 - y = 0 \rightarrow y(y-1) = 0$

$y = 0 \rightarrow x = f(0) = g(0) = -2$

$y = 1 \rightarrow x = f(1) = g(1) = 0$

Area: $A = \displaystyle\int_{0}^{1} [f(y) - g(y)]\, dy$

$A = \displaystyle\int_{0}^{1} \left[(y^2 + y - 2) - (2y^2 - 2)\right]\, dy$

$A = \displaystyle\int_{0}^{1} \left[-y^2 + y\right]\, dy$

$A = \left(-\frac{1}{3}\,y^3 + \frac{1}{2}\,y^2\right)\Big|_{0}^{1}$

$A = \frac{1}{6} - 0 = \frac{1}{6} \approx 0.17$ square unit

Example 15:

Find the area of the region bounded by $f(y) = 4y - 8$, $g(y) = y^3 - 8$ between $y = 1$ and $y = 2$

The graph shows three points of intersections:

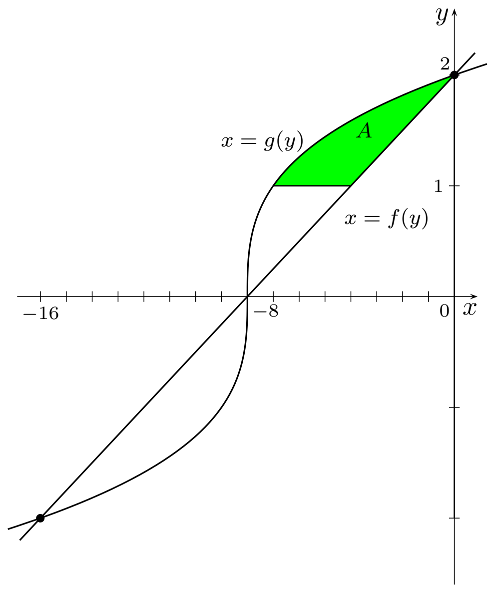
$f(y) = g(y) \implies y^3 - 8 = 4y - 8$

$y^3 - 4y = 0 \rightarrow y(y-2)(y+2) = 0$

$y = -2 \rightarrow x = f(-2) = g(-2) = -16$

$y = 0 \rightarrow x = f(0) = g(0) = -8$

$y = 1 \rightarrow f(1) = -4 \; ; \; g(1) = -7$



Area: $A = \int_1^2 \left[f(y) - g(y)\right] \, dy$

$A = \int_1^2 \left[(4y - 8) - (y^3 - 8)\right] \, dy$

$A = \int_1^2 \left[-y^3 - 4y\right] \, dy$

$A = \left(-\frac{1}{4}y^4 - 2\,y^2\right)\Big|_1^2$

$A = 4 - \frac{7}{4} = \frac{9}{4} = 2.25$ square units