Basic Formulas

Power Rule: $\boxed{kx^n \longrightarrow k\,\dfrac{x^{n+1}}{n+1} + C \;\; ; \;\; n \neq -1}$; this formula does not work for $n = -1$

Logarithmic formula: $\boxed{\dfrac{1}{x} = x^{-1} \longrightarrow \ln|x| + C}$

Exponential formulas: $\boxed{e^x \longrightarrow e^x + C}$

$\boxed{a^x \longrightarrow \dfrac{a^x}{\ln(a)} + C}$

These formulas can be used with the operations of addition and subtraction.

Notation & Symbols of integration

Find the integral of $12x^6$. Instead of this question in words, we use the following:

$$\int (12x^6)\,dx$$

$\int$ is called the integration sign (like S letter with long neck)

$(12x^6)$ the term you find the integral (antiderivative) is in parenthesis

dx is called the differential of x ; also indicates that the answer is in terms of x

the question means find the antiderivative or find the indefinite integral.

Solution : $\displaystyle\int (12x^6)\,dx = 12\,\frac{x^7}{7} + C = \frac{12}{7}\,x^7 + C$

Example 1: Basic Rules

Determine each integral:

(1) $\displaystyle\int 6\sqrt{x}\,dx = \int \left(6x^{1/2}\right)\,dx = 6\,\frac{x^{3/2}}{3/2} + C = 4x^{3/2} + C = 4x\sqrt{x} + C$

(2) $\displaystyle\int 15\,dr = 15r + C$

(3) $\displaystyle\int \frac{5}{x}\,dx = \int \left(5x^{-1}\right)\,dx = 5\,\ln|x| + C$

(4) $\displaystyle\int 3^w\,dw = \frac{3^w}{\ln(3)} + C$

(5) $\displaystyle\int \frac{8}{\sqrt[3]{x}}\,dx = \int \left(8x^{-1/3}\right)\,dx = 8\,\frac{x^{2/3}}{2/3} + C = 12x^{2/3} + C = 12\,\sqrt[3]{x^2} + C$

Example 2: Sum & Difference of terms

To find the integral of a sum or a difference of terms, use appropriate basic formulas for each term.

(a) $\displaystyle\int \left(36x + \frac{x^2}{3}\right)\,dx = \int \left(36x + \frac{1}{3}\,x^2\right)\,dx = 36\,\frac{x^2}{2} + \frac{1}{3}\,\frac{x^3}{3} + C = 18x^2 + \frac{1}{9}\,x^3 + C = 18x + \frac{x^3}{9} + C$

(b) $\displaystyle\int \left(25\,\sqrt[4]{x} - 10\right)\,dx = \int \left(25x^{1/4} - 10\right)\,dx = 25\,\frac{x^{5/4}}{5/4} - 10x + C = 20x^{5/4} - 10x + C = 20x\,\sqrt[4]{x} - 10x + C$

(c) $\displaystyle\int \left[3\,(3^x) - 4x^{-1} + 7\,e^x\right]\,dx = 3\,\frac{3^x}{\ln(3)} - 4\,\ln|x| + 7\,e^x + C$

(d) $\displaystyle\int \left(\frac{4x^3}{5} - \frac{17}{x} + 15\,\sqrt{x}\right)\,dx = \int \left(\frac{4}{5}\,x^3 - 17x^{-1} + 15x^{1/2}\right)\,dx$

$\displaystyle= \frac{4}{5}\,\frac{x^4}{4} - 17\,\ln|x| + 15\,\frac{x^{3/2}}{3/2} + C = \frac{1}{5}\,x^4 - 17\,\ln|x| + 10x^{3/2} + C$

(e) $\displaystyle\int \left(\frac{12}{\sqrt{x}} - \frac{8}{x} + \frac{x}{8} - \frac{6}{x^2}\right)\,dx = \int \left(12x^{-1/2} - 8x^{-1} + \frac{1}{8}\,x - 6x^{-2}\right)\,dx$

$\displaystyle= 12\,\frac{x^{1/2}}{1/2} - 8\,\ln|x| + \frac{1}{8}\,\frac{x^2}{2} - 6\frac{x^{-1}}{-1} + C = 24\,\sqrt{x} - 8\,\ln|x| + \frac{1}{16}\,x^2 + \frac{6}{x} + C$

Example 3: Products of terms

To find the integral of a product of terms, must first multiply, reduce if necessary,
then use appropriate basic formulas for each term.

(a) $\displaystyle\int 3x\,\left(2 - x^2\right)\,dx = \int \left(6x - 12x^3\right)\,dx = 6\,\frac{x^2}{2} - 12\,\frac{x^4}{4} + C = 3x^2 - 3x^4 + C$

(b) $\displaystyle\int 6x^{-2}\left(4x + \frac{x^2}{2}\,e^x\right)\,dx = \int \left(24x^{-1} + 3\,e^x\right)\,dx = 24\,\ln|x| + 3\,e^x + C$

(c) $\displaystyle\int (3x + 7)\,dx = \int \left(9x^2 + 42x + 49\right)\,dx = 9\,\frac{x^3}{3} + 42\,\frac{x^2}{2} + 49x + C = 3x^3 + 21x^2 + 49x + C$

(d) $\displaystyle\int \left(x^3 + 1\right)\,(2 - x)\,dx = \int \left(-x^4 + 2x^3 - x + 2\right)\,dx$

$\displaystyle= -\frac{x^5}{5} + 2\,\frac{x^4}{4} - \frac{x^2}{2} + 2x + C = -\frac{1}{5}\,x^5 + \frac{1}{2}\,x^4 - \frac{1}{2}\,x^2 + 2x + C$

(e) $\displaystyle\int (\sqrt{x} + 6x)\,(2 - x)\,dx = \int \left(2\,\sqrt{x} - x + 18x - 6x^{3/2}\right)\,dx = \int \left(2x^{1/2} - x + 18x - 6x^{3/2}\right)\,dx$

$\displaystyle= \int \left(2\,\sqrt{x} + 17x - 6x^{3/2}\right)\,dx = 2\,\frac{x^{3/2}}{3/2} - 17\,\frac{x^2}{2} - 6\,\frac{x^{5/2}}{5/2} + C = \frac{4}{3}\,x^{3/2} + \frac{17}{2}\,x^2 - \frac{12}{5}\,x^{5/2} + C$

Example 4: Quotient of terms

To find the integral of a quotient of terms, must first divide, reduce if necessary, then use appropriate basic formulas for each term.

(a) $\displaystyle\int \frac{6x^2 - 8x^4 + 2}{x^3}\,dx = \int \left(6x^2 - 8x^4 + 2\right) x^{-3}\,dx = \int \left(6x^{-1} - 8x + 2x^{-3}\right)\,dx$

$\displaystyle = 6\,\ln|x| - 8\,\frac{x^2}{2} + 2\,\frac{x^{-2}}{-2} + C = 6\,\ln|x| - 4x^2 - x^{-2} + C = 6\,\ln|x| - 4x^2 - \frac{1}{x^2} + C$

(b) $\displaystyle\int \frac{12x^4 - 9x\,(4^x)}{3x}\,dx = \int \left[\frac{12x^4}{3x} - \frac{9x\,(4^x)}{3x}\right]\,dx = \int \left[4x^3 - 3\,(4^x)\right]\,dx$

$\displaystyle = 4\,\frac{x^4}{4} - 3\,\frac{4^x}{\ln(4)} + C = x^4 - 3\,\frac{4^x}{\ln(4)} + C$

(c) $\displaystyle\int \frac{15x^2 - 6x}{\sqrt{x}}\,dx = \int \left(15x^2 - 6x\right) x^{-1/2}\,dx = \int \left(15x^{3/2} - 6x^{1/2}\right)\,dx = 15\,\frac{x^{5/2}}{5/2} - 6\,\frac{x^{3/2}}{3/2} + C$

$\displaystyle = 6x^{5/2} - 4x^{3/2} + C = 6\,\sqrt{x^5} - 4\,\sqrt{x^3} + C = 6x^2\,\sqrt{x} - 4x\,\sqrt{x} + C$

(d) $\displaystyle\int \frac{7x^2 - 9x^3\,e^x + 5}{x^3}\,dx = \int \left(7x^2 - 9x^3\,e^x + 5\right)\,x^{-3}\,dx = \int \left(7x^{-1} - 9\,e^x + 5x^{-3}\right)\,dx$

$\displaystyle = 7\,\ln|x| - 9\,e^x + 5\,\frac{x^{-2}}{-2} + C = 7\,\ln|x| - 9\,e^x - \frac{5}{2}\,x^{-2} + C = 7\,\ln|x| - 9\,e^x - \frac{5}{2\,x^2} + C$

(e) $\displaystyle\int \frac{x^2 + x - 6}{x + 3}\,dx = \int \frac{(x+3)(x-2)}{x+3}\,dx = \int (x-2)\,dx = \frac{x^2}{2} - 2x + C = \frac{1}{2}\,x^2 - 2x + C$

Example 5: Determine each integral

(a) $\int \left[2x^2(x-3)+4\,e^x\right]\,dx = \int \left[2x^3-6x^2+4\,e^x\right]\,dx = 2\,\frac{x^4}{4}-6\,\frac{x^3}{3}+4\,e^x+C = \frac{1}{2}\,x^4-2x^3+4\,e^x+C$

(b) $\int \left[5^x-\frac{5}{x}-\frac{x}{5}+x^5\right]\,dx = \int \left[5^x-5x^{-1}-\frac{1}{5}\,x+x^5\right]\,dx = \frac{5^x}{\ln(5)}-5\,\ln|x|-\frac{1}{5}\,\frac{x^2}{2}+\frac{x^6}{6}+C$

$= \frac{5^x}{\ln(5)}-5\,\ln|x|-\frac{1}{10}\,x^2+\frac{1}{6}\,x^6+C$

(c) $\int \frac{52\,\sqrt[3]{x}+25\,\sqrt{x}}{2\,\sqrt[4]{x}}\,dx = \int \left(\frac{52x^{1/3}}{2x^{1/4}}+\frac{25x^{1/2}}{2x^{1/4}}\right)\,dx = \int \left(26x^{1/12}+\frac{25}{2}\,x^{1/4}\right)\,dx$

$= 26\,\frac{x^{13/12}}{13/12}+\frac{25}{2}\,\frac{x^{5/4}}{5/4}+C = 24x^{13/12}+10x^{5/4}+C$

(d) $\int \frac{x^2\,8^x-12+4(x^2-3)}{x^2}\,dx = \int \left(x^2\,8^x-12+4x^2-12\right)\,x^{-2}\,dx = \int \left(8^x+4-24x^{-2}\right)\,dx$

$= \frac{8^x}{\ln(8)}+4x-24\,\frac{x^{-1}}{-1}+C = \frac{8^x}{\ln(8)}+4x+\frac{24}{x}+C$

Example 6: Indefinite integral with condition

The problem gives the first derivative of $f(x)$ with a given condition.
To solve for $f(x)$, find the integral and use the condition to solve for C.

Problem Given $f'(x) = 24\,\sqrt{x}$ with condition $f(4) = 100$. Find $f(x)$

Solution $f(x) = \int \left(24x^{1/2}\right)\,dx = 24\,\frac{x^{3/2}}{3/2}+C = 16x^{3/2}+C$

replace $x = 4$ and $f(4) = 100$ to solve for C:

$100 = 16(4)^{3/2}+C \longrightarrow C = -28$

$f(x) = 16x^{3/2}-28$

Example 7: Marginal Cost

The problem gives the marginal cost with a condition.
To solve for the cost $C(x)$, find the integral and use the condition to solve for k.

Problem Given marginal cost $C'(x) = 1000x + 50\,e^x$ with fixed cost \$300. Find $C(x)$

Solution $C(x) = \int (1000x + 50\,e^x)\,dx = 500x^2 + 50\,e^x + k$

replace $x = 0$ and $C = 300$ to solve for k:

$300 = 500(0)^2 + 50\,e^0 + k \longrightarrow k = 250$

$C(x) = 500x^2 + 50\,e^x + 250$

Example 8: Marginal revenue

The problem gives the marginal revenue of $R(x)$ with the obvious condition (i.e $x = 0$, $R = 0$).
To solve for $R(x)$, find the integral and use the condition to solve for k.

Problem Given the marginal revenue $R'(x) = 50 + 36x^2 + 40x^3$. Find the revenue $R(x)$ and the demand p

Solution $R(x) = \int \left(50 + 36x^2 + 40x^3\right)\,dx = 50x + 12x^3 + 10x^4 + k$

replace $x = 0$ and $R = 0$ to solve for k:

$0 = 50(0) + 12(0)^3 + 10(0)^4 + k \longrightarrow k = 0$

$R(x) = 50x + 12x^3 + 10x^4$

Example 9: Indefinite integral with conditions

The problem gives the second derivative of $f''(x)$ with 2 given conditions.
To solve for $f(x)$, find the integral twice and use the conditions to solve 2 constants of integration.

Problem Given $f''(x) = 60x - 120x^2$ with conditions $f'(1) = f(1) = 5$. Find $f(x)$

Solution $f'(x) = \int \left(60x - 120x^2\right)\, dx = 30x^2 - 40x^3 + C_1$

replace $x = 1$ and $f'(1) = 5$ to solve for C_1:

$5 = 30(1)^2 - 40(1)^3 + C_1 \longrightarrow C_1 = 15 \longrightarrow f'(x) = 30x^2 - 40x^3 + 15$

$f(x) = \int \left(30x^2 - 40x^3 + 15\right)\, dx = 10x^3 - 10x^4 + 15x + C_2$

replace $x = 1$ and $f(1) = 5$ to solve for C_2: $5 = 10(1)^3 - 10(1)^4 + 15(1) + C_2 \longrightarrow C_2 = -10$

$f(x) = 10x^3 - 10x^4 + 15x - 10$

Example 10: Indefinite integral with conditions

The problem gives the second derivative of $f''(x)$ with 2 given conditions.
To solve for $f(x)$, find the integral twice and use the conditions to solve 2 constants of integration.

Problem Given $f''(x) = 54 - 30\sqrt{x}$ with conditions $f'(1) = 40$, $f(1) = 30$. Find $f(x)$

Solution $f'(x) = \int \left(54 - 30x^{1/2}\right)\, dx = 54x - 20x^{3/2} + C_1$

replace $x = 1$ and $f'(1) = 40$ to solve for C_1:

$40 = 54(1) - 20(1)^{3/2} + C_1 \longrightarrow C_1 = 6 \longrightarrow f'(x) = 54x - 20x^{3/2} + 6$

$f(x) = \int \left(54x - 20x^{3/2} + 6\right)\, dx = 27x^2 - 8x^{5/2} + 6x + C_2$

replace $x = 1$ and $f(1) = 30$ to solve for C_2: $30 = 27(1)^2 - 8(1)^{5/2} + 6(1) + C_2 \longrightarrow C_2 = 5$

$f(x) = 27x^2 - 8x^{5/2} + 6x + 5$