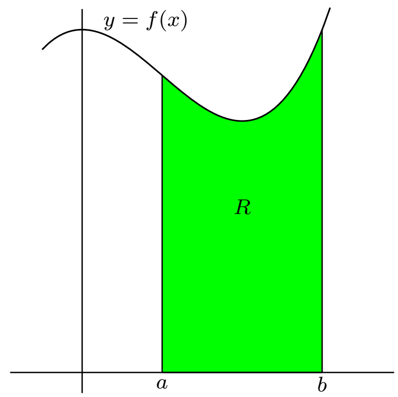
Definite Integral

Let $f(x)$ be a continuous function and

$f(x) > 0$ on a closed interval $[a, b]$ as shown

on the graph.



Then the region R has Area $= \int_a^b f(x)\,dx$

This integral is called the definite integral from a to b.

a is the lower limit of integration and

b is the upper limit of integration.

To evaluate the definite integral, follow the procedure:

(1) find the antiderivative of $f(x)$, let's call it $F(x)$

(2) replace the upper limit b in $F(x)$, we get $F(b)$

(3) replace the lower limit a in $F(x)$, we get $F(a)$

(4) subtract : $F(b) - F(a)$

(5) the answer is a positive real number.

Fundamental Theorem of Calculus

Let $y = f(x)$ be a continuous function on a closed interval $[a, b]$ and let the antiderivative of $f(x)$ be $F(x)$

Then the fundamental theorem of Calculus states: $\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$

Note:

The definite integral does not represent area.

The properties of the definite integral are listed below.

Example 1:

Evaluate $\displaystyle\int_3^6 4x^2\,dx$

$$\int 4x^2\,dx = 4\,\frac{x^3}{3} + C = \frac{4}{3}\,x^3 + C \longrightarrow \int_3^6 4x^2\,dx = \frac{4}{3}\,x^3\Big|_3^6 = \left[\frac{4}{3}\,(6)^3\right] - \left[\frac{4}{3}\,(3)^3\right] = 288 - 36 = 252$$

Example 2:

Evaluate $\displaystyle\int_2^4 (4-x)\,dx$

$$\int (4-x)\,dx = 4x - \frac{x^2}{2} + C = 4x - \frac{1}{2}\,x^2 + C$$

$$\int_2^4 (4-x)\,dx = 4x - \frac{1}{2}\,x^2\Big|_2^4 = \left[4(4) - \frac{1}{2}\,(4)^2\right] - \left[4(2) - \frac{1}{2}\,(2)^2\right] = 8 - 6 = 2$$

Example 3:

Evaluate $\displaystyle\int_0^2 (-x^3)\,dx$

$$\int (-x^3)\,dx = -\frac{x^4}{4} + C = -\frac{1}{4}\,x^4 + C$$

$$\int_0^2 (-x^3)\,dx = -\frac{1}{4}\,x^4\Big|_0^2 = \left[-\frac{1}{4}\,(2)^4\right] - \left[-\frac{1}{4}\,(0)^4\right] = -4 + 0 = -4$$

Note: Definite integrals may give negative answers.

Example 4:

Evaluate $\displaystyle\int_0^2 (x^3 - x)\,dx$

$$\int (x^3 - x)\,dx = \frac{x^4}{4} - \frac{x^2}{2} + C = \frac{1}{4}\,x^4 - \frac{1}{2}\,x^2 + C$$

$$\int_0^2 (x^3 - x)\,dx = \frac{1}{4}\,x^4 - \frac{1}{2}\,x^2\Big|_0^2 = \left[\frac{1}{4}\,(2)^4 - \frac{1}{2}\,(2)^2\right] - \left[\frac{1}{4}\,(0)^4 - \frac{1}{2}\,(0)^2\right] = 2 - 0 = 2$$

Properties of the Definite Integral

Let f and g be continuous functions on $[a, b]$.

(1) $\int_a^b k\,dx = k(b-a)$ (k is a constant)

(2) $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$

(3) $\int_a^b \left[f(x) \pm g(x)\right]dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

(4) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad ; \quad a < c < b$

(5) $\int_a^a f(x)\,dx = 0$

(6) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(7) If $f(x) > g(x)$ on $[a, b]$, then $\int_a^b f(x)\,dx > \int_a^b g(x)\,dx$

(8) If $f(x) < g(x)$ on $[a, b]$, then $\int_a^b f(x)\,dx < \int_a^b g(x)\,dx$

(9) The average value of $f(x)$ on $[a, b]$ is $\overline{f} = f_{av} = \frac{1}{b-a}\int_a^b f(x)\,dx$

(10) If $f(x)$ is an even continuous function, then $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$

(11) If $f(x)$ is an odd continuous function, then $\int_{-a}^a f(x)\,dx = 0$

Example 5:

Given f is an even function and $\displaystyle\int_0^3 f(x)\,dx = 6$, evaluate:

(a) $\displaystyle\int_{-3}^0 f(x)\,dx$; (b) $\displaystyle\int_{-3}^3 f(x)\,dx$; (c) $\displaystyle\int_3^0 f(x)\,dx$; (d) $\displaystyle\int_0^3 \big[6 - f(x)\big]\,dx$

Solution:

(a) since f is an even function, then $\displaystyle\int_{-3}^0 f(x)\,dx = \int_0^3 f(x)\,dx = 6$

(b) since f is an even function, then $\displaystyle\int_{-3}^3 f(x)\,dx = 2\int_0^3 f(x)\,dx = 2(6) = 12$

(c) by property, $\displaystyle\int_3^0 f(x)\,dx = -\int_0^3 f(x)\,dx = -6$

(d) by property, $\displaystyle\int_0^3 \big[6 - f(x)\big]\,dx = \int_0^3 6\,dx - \int_0^3 f(x)\,dx$

where $\displaystyle\int_0^3 6\,dx = 6(3-0) = 18$ and $\displaystyle -\int_0^3 f(x)\,dx = -6$

therefore $\displaystyle\int_0^3 \big[6 - f(x)\big]\,dx = 18 - 6 = 12$

Example 6:

Given $\displaystyle\int_{-1}^2 f(x)\,dx = 7$ and $\displaystyle\int_{-1}^5 f(x)\,dx = 12$

evaluate: $\displaystyle\int_2^5 f(x)\,dx$

Solution:

by property, $\displaystyle\int_{-1}^5 f(x)\,dx = \int_{-1}^2 f(x)\,dx + \int_2^5 f(x)\,dx$

$$\int_{-1}^5 f(x)\,dx - \int_{-1}^2 f(x)\,dx = \int_2^5 f(x)\,dx$$

$$\int_2^5 f(x)\,dx = 12 - 7 = 5$$

Example 7:

Given f is an odd function and $\displaystyle\int_{-2}^{0} f(x)\,dx = 4$, evaluate:

(a) $\displaystyle\int_{0}^{2} f(x)\,dx$; (b) $\displaystyle\int_{-2}^{2} f(x)\,dx$; (c) $\displaystyle\int_{0}^{-2} f(x)\,dx$; (d) $\displaystyle\int_{-2}^{0} \left[-2\,f(x)\right]dx$

Solution:

(a) since f is an odd function, then $\displaystyle\int_{0}^{2} f(x)\,dx = -\int_{-2}^{0} f(x)\,dx = -4$

(b) since f is an odd function, then $\displaystyle\int_{-2}^{2} f(x)\,dx = 0$

(c) by property, $\displaystyle\int_{0}^{-2} f(x)\,dx = -\int_{-2}^{0} f(x)\,dx = -4$

(d) by property, $\displaystyle\int_{-2}^{0} \left[-2\,f(x)\right]dx = -2\int_{-2}^{0} f(x)\,dx = -2(4) = -8$

Example 8:

Find the average value of $f(x) = (x-2)^3$ over $[2,4]$.

Then find x value(s) in the interval such that f is equal to $\overline{f}$.

Solution

The average value of f is $\displaystyle\overline{f} = \frac{1}{b-a}\int_{a}^{b} f(x)\,dx$

$$\overline{f} = \frac{1}{4-2}\int_{2}^{4}(x-2)^3\,dx = \frac{1}{2}\int_{2}^{4}(x-2)^3\,dx$$

The integral is solved by substitution: let $u = x-2 \rightarrow du = dx$

the integral becomes $\displaystyle\int u^3\,du = \frac{u^4}{4} + C$ and the answer is: $\displaystyle\frac{(x-2)^4}{4} + C$

$$\overline{f} = \frac{1}{2}\left[\frac{(x-2)^4}{4}\right]\Bigg|_{2}^{4} = \frac{1}{2}\left[\frac{(4-2)^4}{4}\right] - \frac{1}{2}\left[\frac{(2-2)^4}{4}\right] = 4 - 0 = 4$$

make $f = \overline{f} \rightarrow (x-2)^3 = 4 \rightarrow x-2 = \sqrt[3]{4} \approx 1.5874 \rightarrow x \approx 3.587 \ \in \ [2,4]$

Example 9:

Find the average value of $f(x) = e^{-0.4}$ over $[0, 2]$.

Then find x value(s) in the interval such that f is equal to $\overline{f}$.

Solution

The average value of f is $\overline{f} = \dfrac{1}{b-a} \displaystyle\int_a^b f(x)\,dx$

$$\overline{f} = \frac{1}{2-0} \int_0^2 e^{-0.4x}\,dx = \frac{1}{2} \int_0^2 e^{-0.4x}\,dx$$

The integral is solved by formula: $\displaystyle\int e^{k\,x}\,du = \frac{1}{k}\,e^{k\,x} + C$

$$\overline{f} = \frac{1}{2}\left[\frac{1}{-0.4}\,e^{-0.4x}\right]\Bigg|_0^2 = \left[\frac{-5}{4}\,e^{-0.8}\right] - \left[\frac{-5}{4}\,e^{0}\right] \approx 0.688339$$

make $f = \overline{f} \to e^{-0.4x} = 0.688339 \to -0.4x = \ln(0.688339) \approx -0.3734 \to x \approx 0.934 \in [0, 2]$

Example 10:

Given $\displaystyle\int_1^4 f(x)\,dx = 5$ and $\displaystyle\int_1^4 g(x)\,dx = 2$

evaluate: $\displaystyle\int_1^4 \big[3\,f(x) - 2\,g(x) + 5\big]\,dx$

Solution:

by property, $\displaystyle\int_1^4 3\,f(x)\,dx - \int_1^4 2\,g(x)\,dx + \int_1^4 5\,dx$

$$3\int_1^4 f(x)\,dx - 2\int_1^4 g(x)\,dx + 5(4-1)$$

replacing, we get: $\displaystyle\int_1^4 \big[3\,f(x) - 2\,g(x) + 5\big]\,dx = 3(5) - 2(2) + 5(3) = 26$

Example 11:

Given $\int_{-1}^{1} f(x)\,dx = 2$ and $\int_{-1}^{3} f(x)\,dx = 5$

evaluate: $\int_{1}^{3} \big[4\,f(x) + 7\big]\,dx$

Solution:

by property, $\int_{-1}^{3} f(x)\,dx = \int_{-1}^{1} f(x)\,dx + \int_{1}^{3} f(x)\,dx$ then $\int_{1}^{3} f(x)\,dx = 5 - 2 = 3$

$$\int_{1}^{3} \big[4\,f(x) + 7\big]\,dx = 4\int_{1}^{3} f(x)\,dx + \int_{1}^{3} 7\,dx$$

replacing, we get: $\int_{1}^{3} \big[4\,f(x) + 7\big]\,dx = 4(3) + 7(3 - 1) = 26$

Example 12:

Given $\int_{-2}^{2} f(x)\,dx = 7$ and $\int_{5}^{-2} f(x)\,dx = -10$

evaluate: $\int_{2}^{5} 3\,f(x)\,dx$

Solution:

by property, $\int_{-2}^{5} f(x)\,dx = -\int_{5}^{-2} f(x)\,dx = 10$

$$\int_{-2}^{5} f(x)\,dx = \int_{-2}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx$$

then $\int_{2}^{5} f(x)\,dx = 10 - 7 = 3$

therefore $\int_{2}^{5} 3\,f(x)\,dx = 3\int_{2}^{5} f(x)\,dx = 3(3) = 9$