L'Hôpital's Rule - Indeterminate Forms

Statement & Condition to apply L'Hôpital's Rule:

Given $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ or $\dfrac{-\infty}{\infty}$ or $\dfrac{\infty}{-\infty}$

these answers are called indeterminate forms

Then L'Hôpital's Rule is applied as follows:

$$\boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}}$$

Repeat the Rule until the indeterminate form is gone.

Example 1:

Given $\lim\limits_{x\to -1} \dfrac{x^2 - x - 2}{x^3 + 1} = \dfrac{(-1)^2 - (-1) - 2}{(-1)^3 + 1} = \dfrac{0}{0}$, indeterminate form; L'Hôpital's Rule can be used:

$$\lim_{x\to -1} \frac{2x - 1}{3x^2} = \frac{2(-1) - 1}{3(-1)^2} = -1$$

Example 2:

Given $\lim\limits_{x\to\infty} \dfrac{\sqrt{x+4}}{x} = \dfrac{\infty}{\infty}$, indeterminate form; L'Hôpital's Rule can be used:

$$\lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x+4}}}{1} = \lim_{x\to\infty} \frac{1}{2\sqrt{x+4}} = \frac{1}{2\sqrt{\infty+4}} = 0$$

Example 3:

Given $\lim\limits_{x\to\infty} \dfrac{\sqrt{x^2+9}}{2x} = \dfrac{\infty}{\infty}$, indeterminate form; L'Hôpital's Rule can be used:

$\lim\limits_{x\to\infty} \dfrac{\frac{2x}{2\sqrt{x^2+9}}}{2} = \lim\limits_{x\to\infty} \dfrac{x}{2\sqrt{x^2+9}} = \dfrac{\infty}{\infty}$ if you repeat L'Hôpital's Rule, it won't help; the rule fails!

To solve, need to do Algebra work as follows: $\lim\limits_{x\to\infty} \dfrac{\sqrt{x^2+9}}{2x} = \lim\limits_{x\to\infty} \dfrac{x\sqrt{\frac{x^2}{x^2} + \frac{9}{x^2}}}{2x} = \lim\limits_{x\to\infty} \dfrac{\sqrt{1 + \frac{9}{x^2}}}{2}$

$$\lim_{x\to\infty} \frac{\sqrt{1 + \frac{9}{x^2}}}{2} = \frac{\sqrt{1+0}}{2} = \frac{1}{2}$$

Example 4:

Given $\lim\limits_{x\to 0} \dfrac{6\,\sin(5x)}{7\,\sin(2x)} = \dfrac{0}{0}$, indeterminate form; L'Hôpital's Rule can be used:

$$\lim_{x\to 0} \frac{6(5)\,\cos(5x)}{7(2)\,\cos(2x)} = \lim_{x\to 0} \frac{30\,\cos(5x)}{14\,\cos(2x)} = \lim_{x\to 0} \frac{15\,\cos(5x)}{7\,\cos(2x)} = \frac{15\,\cos(0)}{7\,\cos(0)} = \frac{15(1)}{7(1)} = \frac{15}{7}$$

Example 5:

Given $\lim\limits_{x\to 0} \dfrac{\tan(x)+5x}{e^x-1} = \dfrac{\tan(0)+5(0)}{e^0-1} = \dfrac{0}{0}$, indeterminate form; L'Hôpital's Rule can be used:

$$\lim_{x\to 0} \frac{\tan(x)+5x}{e^x-1} = \lim_{x\to 0} \frac{\sec^2(x)+5}{e^x} = \frac{\sec^2(0)+5}{e^0} = \frac{1+5}{1} = 6$$

Example 6:

Given $\lim\limits_{x\to 3} \dfrac{x^2-9}{\ln(x-2)} = \dfrac{(3)^2-9}{\ln(3-2)} = \dfrac{0}{0}$, indeterminate form; L'Hôpital's Rule can be used:

$$\lim_{x\to 3} \frac{x^2-9}{\ln(x-2)} = \lim_{x\to 3} \frac{2x}{\frac{1}{x-2}} = \lim_{x\to 3} 2x(x-2) = 2(3)(3-2) = 6$$

Example 7:

Given $\lim\limits_{x\to \infty} \dfrac{e^{2x}}{x^2+3} = \dfrac{e^{\infty}}{(\infty)^2+3} = \dfrac{\infty}{\infty}$, indeterminate form; L'Hôpital's Rule can be used:

$\lim\limits_{x\to \infty} \dfrac{e^{2x}}{x^2+3} = \lim\limits_{x\to \infty} \dfrac{2\,e^{2x}}{2x} = \lim\limits_{x\to \infty} \dfrac{e^{2x}}{x} = \dfrac{\infty}{\infty}$, indeterminate form; L'Hôpital's Rule can be used again:

$$\lim_{x\to \infty} \frac{e^{2x}}{x} = \lim_{x\to \infty} \frac{2\,e^{2x}}{1} = 2\,e^{2x} = 2\,e^{\infty} = \infty$$

Example 8:

Given $\lim\limits_{x\to \pi} \dfrac{\sin^2(x)}{\cos(x)+1} = \dfrac{\sin^2(\pi)}{\cos(\pi)+1} = \dfrac{0}{0}$, indeterminate form; L'Hôpital's Rule can be used:

$$\lim_{x\to \pi} \frac{2\,\sin(x)\,\cos(x)}{-\sin(x)} = \lim_{x\to \pi} \big(-2\,\cos(x)\big) = -2\,\cos(\pi) = 2$$

Example 9:

Given $\lim_{x\to\infty} \dfrac{2x^4-3x+1}{3x^4-2x+5} = \dfrac{2(\infty)^4-3(\infty)+1}{3(\infty)^4-2(\infty)+5} = \dfrac{\infty}{\infty}$, indeterminate form; L'Hôpital's Rule can be used:

$$\lim_{x\to\infty} \frac{2x^4-3x+1}{3x^4-2x+5} = \lim_{x\to\infty} \frac{8x^3-3}{12x^3-2} = \lim_{x\to\infty} \frac{8(\infty)^3-3}{12(\infty)^3-2} = \frac{\infty}{\infty}$$, indeterminate form;

L'Hôpital's Rule can be used again: $\lim_{x\to\infty} \dfrac{8x^3-3}{12x^3-2} = \lim_{x\to\infty} \dfrac{24x^2}{36x^2} = \dfrac{24}{36} = \dfrac{2}{3}$

Example 10:

Given $\lim_{x\to-2} \dfrac{x^3+8}{x^2-4} = \dfrac{(-2)^3+8}{(-2)^2-4} = \dfrac{0}{0}$, indeterminate form; L'Hôpital's Rule can be used:

$$\lim_{x\to-2} \frac{x^3+8}{x^2-4} = \lim_{x\to-2} \frac{3x^2}{2x} = \lim_{x\to-2} \frac{3x}{2} = \frac{3(-2)}{2} = -3$$

Example 11:

Given $\lim_{x\to-1} \dfrac{\tan(x+1)-2x-2}{\cos(x+1)+x} = \dfrac{\tan(0)-2(-1)-2}{\cos(0)-1} = \dfrac{0}{0}$, indeterminate form; L'Hôpital's Rule can be used:

$$\lim_{x\to-1} \frac{\tan(x+1)-2x-2}{\cos(x+1)+x} = \lim_{x\to-1} \frac{\sec^2(x+1)-2}{-\sin(x+1)+1} = \frac{\sec^2(0)-2}{-\sin(0)+1} = \frac{1-2}{1} = -1$$

Example 12:

Given $\lim_{x\to\infty} \dfrac{4\,e^{2x}}{3^x} = \dfrac{\infty}{\infty}$, indeterminate form; L'Hôpital's Rule can be used:

$\lim_{x\to\infty} \dfrac{4\,e^{2x}}{3^x} = \lim_{x\to\infty} \dfrac{8\,e^{2x}}{3^x\,\ln(3)} = \dfrac{\infty}{\infty}$ if you repeat L'Hôpital's Rule, it won't help; the rule fails!

To solve, compare the exponentials as follows: when $x\to\infty$ then $4\,e^{2x} < 3^x$ and $\lim_{x\to\infty} \dfrac{4\,e^{2x}}{3^x} = 0$