Definition & Types of Improper Integrals

Improper integral is a definite integral: $\int_a^b f(x)\,dx$

with 2 types of conditions shown as follows:

Type 1: at least one of the limits of integration is undefined like $\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$

Type 2: one of the limits of integration makes $f(x)$ undefined

let $x = b$, $f(b) =$ undefined, then $\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$

if $x = a$, $f(a) =$ undefined, then $\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$

Two possible answers:

convergent if the result is a real number.

divergent if the result is undefined.

Example 1:

Evaluate $\int_1^\infty \frac{4}{x^3}\,dx$ it is an improper integral.

Type 1: $\int_1^\infty 4x^{-3}\,dx = \lim_{t\to\infty} \int_1^t 4x^{-3}\,dx$

$$\lim_{t\to\infty} \int_1^t 4x^{-3}\,dx = \lim_{t\to\infty} \left[4\,\frac{x^{-2}}{-2}\right]_1^t = \lim_{t\to\infty} \left[\frac{-2}{x^2}\right]_1^t = \lim_{t\to\infty} \left[\frac{-2}{t^2} - \frac{-2}{1^2}\right] = \frac{-2}{\infty} + 2 = 0 + 2 = 2 \text{ convergent}$$

Example 2:

Evaluate $\int_0^2 \frac{4}{x^5}\,dx$ it is an improper integral.

Type 2: at $x = 0$, $f(0) = \frac{4}{(0)^5} =$ undefined

then $\int_0^2 4x^{-5}\,dx = \lim_{t\to 0^+} \int_t^2 4x^{-5}\,dx$

$$\lim_{t\to 0^+} \int_t^2 4x^{-5}\,dx = \lim_{t\to 0^+} \left[4\,\frac{x^{-4}}{-4}\right]_t^2 = \lim_{t\to 0^+} \left[\frac{-1}{x^4}\right]_t^2 = \lim_{t\to 0^+} \left[\frac{-1}{(2)^4} - \frac{-1}{t^4}\right] = \frac{-1}{16} - \frac{-1}{0} = \frac{-1}{16} + \infty = +\infty \text{ divergent}$$

Example 3:

Evaluate $\displaystyle\int_{-\infty}^{0} \frac{1}{\sqrt{4-x}}\,dx$ it is an improper integral.

Type 1: $\displaystyle\int_{-\infty}^{0} (4-x)^{-1/2}\,dx = \lim_{t\to-\infty} \int_{t}^{0} (4-x)^{-1/2}\,dx$

use substitution: $u = 4-x$, $du = -dx \to -du = dx$

replacing: $\displaystyle\int -u^{-1/2}\,du = -\frac{u^{1/2}}{1/2} + C = -2\,u^{1/2} + C$

then $\displaystyle\lim_{t\to-\infty} \int_{t}^{0} (4-x)^{-1/2}\,dx = \lim_{t\to-\infty} \Big[-2\sqrt{4-x}\,\Big]_{t}^{0} = \lim_{t\to-\infty} \Big[\big(-2\sqrt{4-0}\,\big) - \big(-2\sqrt{4-t}\,\big)\Big]$

$= \Big[-2\,(2) + 2\sqrt{+\infty}\Big] = -4 + \infty = +\infty$ divergent

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Example 4:

Evaluate $\displaystyle\int_{1}^{5} \frac{1}{\sqrt{x-1}}\,dx$ it is an improper integral.

Type 2: at $x = 1$, $f(1) = \dfrac{1}{\sqrt{0}} =$ undefined

use substitution: $u = x-1$, $du = dx$

replacing: $\displaystyle\int u^{-1/2}\,du = \frac{u^{1/2}}{1/2} + C = 2\,u^{1/2} + C$

then $\displaystyle\int_{1}^{5} (x-1)^{-1/2}\,dx = \lim_{t\to1^+} \int_{t}^{5} (x-1)^{-1/2}\,dx$

$\displaystyle\lim_{t\to1^+} \int_{t}^{5} (x-1)^{-1/2}\,dx = \lim_{t\to1^+} \Big[2\sqrt{x-1}\,\Big]_{t}^{5} = \lim_{t\to1^+} \Big[\big(2\sqrt{5-1}\,\big) - \big(2\sqrt{t-1}\,\big)\Big]$

$= \Big[2\,(2) - 2\sqrt{0}\Big] = 4 - 0 = 4$ convergent

Example 5:

Evaluate $\displaystyle\int_0^{\infty} e^{-3x}\,dx$ it is an improper integral.

Type 1: $\displaystyle\int_0^{\infty} e^{-3x}\,dx = \lim_{t\to\infty} \int_0^{t} e^{-3x}\,dx$

use exponential formula: $\displaystyle\int e^{kx}\,dx = \tfrac{1}{k}\,e^{kx} + C$

then $\displaystyle\lim_{t\to\infty} \int_0^{t} e^{-3x}\,dx = \lim_{t\to\infty} \left[-\frac{1}{3}\,e^{-3x}\right]_0^t = \lim_{t\to\infty} \left[\left(-\frac{1}{3}\,e^{-3t}\right) - \left(-\frac{1}{3}\,e^{0}\right)\right]$

$\displaystyle = \left[-\frac{1}{3}\,e^{-\infty} + \frac{1}{3}\right] = 0 + \frac{1}{3} = \frac{1}{3}$ convergent

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Example 6:

Evaluate $\displaystyle\int_0^{3} \frac{3}{x}\,dx$ it is an improper integral.

Type 2: at $x = 0\,,\ f(0) = \dfrac{3}{0} =$ undefined

use logarithmic formula: $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$

then $\displaystyle\int_0^{3} \frac{3}{x}\,dx = \lim_{t\to 0^+} \int_t^{3} \frac{3}{x}\,dx$

$\displaystyle\lim_{t\to 0^+} \int_t^{3} \frac{3}{x}\,dx = \lim_{t\to 0^+} \Big[3\,\ln|x|\Big]_t^3 = \lim_{t\to 0^+} \Big[3\,\ln(3) - 3\,\ln|t|\Big]$

$\displaystyle = \Big[3\,\ln(3) - 3\,\ln(0^+)\Big] = 3\,\ln(3) - (-\infty) = \infty$ divergent

Example 7:

Evaluate $\displaystyle\int_{-8}^{0} \frac{4}{\sqrt[3]{x^2}}\,dx$ it is an improper integral.

Type 2: at $x = 0\,,\ f(0) = \dfrac{4}{0} =$ undefined

then $\displaystyle\int_{-8}^{0} 4x^{-2/3}\,dx = \lim_{t\to 0^-} \int_{-8}^{t} 4x^{-2/3}\,dx$

$$\lim_{t\to 0^-} \int_{-8}^{t} 4x^{-2/3}\,dx = \lim_{t\to 0^-} \left[4\,\frac{x^{1/3}}{1/3}\right]_{-8}^{t} = \lim_{t\to 0^-} \left[12x^{1/3}\right]_{-8}^{t} = \lim_{t\to 0^-} \left[12t^{1/3} - 12(-8)^{1/3}\right]$$

$= \left[12(0) - 12(-2)\right] = 0 + 24 = 24$ convergent