## Linear Denominator

Use log formula $\boxed{\int \frac{k}{ax+b}\,dx = \frac{k}{a}\ln|ax+b| + C}$ shown in previous tutorial.

Given $\frac{P(x)}{Q(x)}$ where $Q(x)$ is a product of 2 linear factors.

The objective is to split the quotient into 2 partial fractions:

Example: $\frac{P(x)}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{cx+d}$ where $A$ and $B$ are constants (real numbers)

( $A$ and $B$ are one degree less than degree of denominator )

Also remember to verify if you must divide before setting the partial fractions.

must divide if (degree of numerator) $P(x) \geq Q(x)$ (degree of denominator).

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## Partial Fractions Procedure

Given $\frac{2x+9}{(x+2)(x-3)}$ , split the quotient into partial fractions.

The degree of the numerator is 1 and the degree of the denominator is 2 , therefore can't divide.

The denominator has 2 linear factors, so 2 partial fractions are set up as follows:

$$\frac{2x+9}{(x+2)(x-3)} = \frac{A}{x+2} + \frac{B}{x-3} = \frac{A(x-3)+B(x+2)}{(x+2)(x-3)}$$

Once the common denominator is done, must find $A$ and $B$ values by equating: $2x+9 = A(x-3)+B(x+2)$

### Method 1:

To solve for $A$ : replace $x = -2 \rightarrow 5 = -5A \rightarrow A = -1$

To solve for $B$ : replace $x = 3 \rightarrow 15 = 5B \rightarrow B = 3$

Therefore the partial fractions are: $\frac{2x+9}{(x+2)(x-3)} = \frac{-1}{x+2} + \frac{3}{x-3}$

Partial Fractions Procedure

Method 2:

Build a system of equations with 2 variables $A$ and $B$ using $2x+9 = A(x-3)+B(x+2) = Ax-3A+Bx+2B$

equating $x$ terms: $2 = A+B$ ; equating constants: $9 = -3A+2B$

multiply the first by 3 and add it to the second, we get: $\left.\begin{array}{rr} 6 = & 3A+3B \\ 9 = & -3A+2B \end{array}\right\}$ $15 = 5B \to 3 = B$

replace $B=3$ in the first: $2 = A+3 \to -1 = A$

$$\frac{2x+9}{(x+2)(x-3)} = \frac{-1}{x+2} + \frac{3}{x-3}$$

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Example 1:

Determine $\displaystyle\int \frac{2x+9}{(x+2)(x-3)}\,dx$

by partial fractions procedure, we have: $\displaystyle\int \frac{2x+9}{(x+2)(x-3)}\,dx = \int \left[\frac{-1}{x+2} + \frac{3}{x-3}\right] dx$

Use log formula $\displaystyle\int \frac{k}{ax+b}\,dx = \frac{k}{a}\ln|ax+b| + C$ shown in previous tutorial.

$$\int \frac{2x+9}{(x+2)(x-3)}\,dx = \int \left[\frac{-1}{x+2} + \frac{3}{x-3}\right] dx = -\ln|x+2| + 3\ln|x-3| + C$$

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Example 2:

Determine $\displaystyle\int \frac{8-10x}{(2x-1)(2-x)}\,dx$ ; can't divide and must split the quotient into 2 partial fractions.

$\displaystyle\frac{8-10x}{(2x-1)(2-x)} = \frac{A}{2x-1} + \frac{B}{2-x}$ common denominator gives: $8-10x = A(2-x)+B(2x-1)$

Method 1:

To solve for $A$ : replace $x = -\frac{1}{2}$
$3 = \frac{3}{2}A \to A = 2$

To solve for $B$ : replace $x = 2$
$-12 = 3B \to B = -4$

Method 2:

equating $x$ terms: $-10 = -A+2B$
equating constants: $8 = 2A-B$

multiply the first by 2 and add it to the second,
we get: $\left.\begin{array}{rr} -20 = & -2A+4B \\ 8 = & 2A-B \end{array}\right\}$
$-12 = 3B \to -4 = B$

replace $B=-4$ in the first:
$-10 = -A+2(-4) \to 2 = A$

$$\int \frac{8-10x}{(2x-1)(2-x)}\,dx = \int \left[\frac{2}{2x-1} + \frac{-4}{2-x}\right] dx = \frac{2}{2}\ln|2x-1| + \frac{-4}{-1}\ln|2-x| + C$$

$$= \ln|2x-1| + 4\ln|2-x| + C$$

Example 3:

Determine $\displaystyle\int \frac{x^3 - x^2 + 3x + 12}{(x-1)(x+2)}\,dx = \int \frac{x^3 - x^2 + 3x + 12}{x^2 + x - 2}\,dx$

must divide and then split the remaining quotient into 2 partial fractions.

Long Division:

$$\begin{array}{r|l} & \quad\quad\quad\quad x - 2 \\ x^2 + x - 2 & x^3 - x^2 + 3x + 12 \\ & -(x^3 + x^2 - 2x) \\ & \quad -2x^2 + 5x + 12 \\ & \quad -(-2x^2 - 2x + 4) \\ & \quad\quad\quad 7x + 8 \end{array}$$

Thus

$$\int \frac{x^3 - x^2 + 3x + 12}{(x-1)(x+2)}\,dx$$

$$= \int \left( x - 2 + \frac{7x+8}{(x-1)(x+2)} \right) dx$$

$\displaystyle\frac{7x+8}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$ common denominator gives: $7x + 8 = A(x+2) + B(x-1)$

Method 1:

To solve for $A$ : replace $x = 1$
$15 = 3A \to A = 5$

To solve for $B$ : replace $x = -2$
$-6 = -3B \to B = 2$

Method 2:

equating $x$ terms: $7 = A + B$
equating constants: $8 = 2A - B$

add the 2 equations,
we get: $\left.\begin{array}{rl} 7 = & A + B \\ 8 = & 2A - B \end{array}\right\}$
$15 = 3A \to 5 = A$

replace $A = 5$ in the first:
$7 = 5 + B \to 2 = B$

$$\int \frac{x^3 - x^2 + 3x + 12}{(x-1)(x+2)}\,dx = \int \left[ x - 2 + \frac{7x+8}{(x-1)(x+2)} \right] dx$$

$$= \int \left[ x - 2 + \frac{5}{x-1} + \frac{2}{x+2} \right] dx = \frac{1}{2}x^2 - 2x + 5\ \ln|x-1| + 2\ \ln|x+2| + C$$

Repeated Linear Denominator

Use log formula $\displaystyle\int \frac{k}{ax+b}\,dx = \frac{k}{a}\ln|ax+b| + C$ shown in previous tutorial.

Given $\dfrac{P(x)}{Q(x)}$ where $Q(x)$ is a product of repeated linear factors.

The objective is to split the quotient into partial fractions:

Example: $\dfrac{P(x)}{(ax+b)^2(cx+d)^3} = \dfrac{A}{ax+b} + \dfrac{B}{(ax+b)^2} + \dfrac{C}{cx+d} + \dfrac{D}{(cx+d)^2} + \dfrac{E}{(cx+d)^3}$

where $A$ , $B$ , $C$ , $D$ and $E$ are constants (real numbers)

Note: 2 repeated linear generates 2 partial fractions and 3 repeated linear generates 3 partial fractions; each repeated linear has increasing powers.

Also remember to verify if you must divide before setting the partial fractions.

must divide if (degree of numerator) $P(x) \geq Q(x)$ (degree of denominator).

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Partial Fractions Procedure

Given $\dfrac{P(x)}{(ax+b)^n}$ , split the quotient into partial fractions.

Assume that the degree of $P(x) <$ the degree of the denominator (which is $n$ ), therefore can't divide.

The denominator has $n$ repeated linear factors, so $n$ partial fractions are set up as follows:

$$\frac{P(x)}{(ax+b)^n} = \frac{A}{ax+b} + \frac{B}{(ax+b)^2} + \frac{C}{(ax+b)^3} + \cdots + \frac{K}{(ax+b)^n}$$

Once the common denominator is done, must find $A$ , $B$ , $C$ , ... , $K$ values.

The first partial fraction, use log formula and the other partial fractions, use substitution: $u = ax + b$

Example 4:

Determine $\int \frac{5x^2-8x+2}{x(x-1)^2}\,dx$ ; can't divide and must split the quotient into partial fractions.

$\frac{5x^2-8x+2}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$ common denominator gives: $5x^2-8x+2 = A(x-1)^2 + Bx(x-1) + Cx$

Method 1:

To solve for $A$ : replace $x=0$
$2 = A \to A = 2$

To solve for $C$ : replace $x=1$
$-1 = C \to C = -1$

To solve for $B$ : replace
$A=2$ , $C=-1$ , $x=2$
$6 = 2 + 2B - 2 \to B = 3$

Method 2:
Using $5x^2-8x+2 = Ax^2 - 2Ax + A + Bx^2 - Bx + Cx$

equating $x^2$ terms: $5 = A + B$
equating $x$ terms: $-8 = -2A - B + C$
equating constants: $2 = A \to A = 2$

replace $A=2$ in first equation, we get: $B=3$

replace $A=2$ , $B=3$ in the second equation, we get:
$C=-1$

$$\int \frac{5x^2-8x+2}{x(x-1)^2}\,dx = \int \left[\frac{2}{x} + \frac{3}{x-1} + \frac{-1}{(x-1)^2}\right] dx$$

use log formula shown in another tutorial for the first 2 fractions: $\int \frac{k}{ax+b}\,dx = \frac{k}{a}\ln|ax+b| + C$

and the last fraction, let: $u = x-1$ and $-du = -dx$ , then we get $\int \frac{-1}{u^2}\,du = \frac{1}{u} + C$ .

Then $\int \frac{5x^2-8x+2}{x(x-1)^2}\,dx = \int \left[\frac{2}{x} + \frac{3}{x-1} + \frac{-1}{(x-1)^2}\right] dx = 2\ln|x| + 3\ln|x-1| + \frac{1}{x-1} + C$