Integration by Parts Formula

The integral is given in the form of a product of 2 expressions $\int u\,.\,dv$
Therefore the formula is derived from the product rule formula as follows:

Product Rule: $\dfrac{d(u\,.\,v)}{dx} = v\,.\,\dfrac{du}{dx} + u\,.\,\dfrac{dv}{dx}$

Differential: $d(u\,.\,v) = v\,.\,du + u\,.\,dv$

Integrate: $\int d(u\,.\,v) = \int v\,.\,du + \int u\,.\,dv$

$$u\,.\,v = \int v\,.\,du + \int u\,.\,dv$$

rewrite: $\boxed{\int u\,.\,dv = u\,.\,v - \int v\,.\,du}$ This is the integration by Parts Formula.

Integration by Parts Procedure

To integrate a product of 2 expressions like $\int u\,.\,dv$

Choose u and dv in such a way that you can find:

(1) du by taking the derivative of u and (2) v by taking the integral of dv.

Pay attention that you must know the integration of dv and also that once you applied the integration by parts formula, you must get the integral of $\int v\,.\,du$ easier to solve than the integral $\int u\,.\,dv$

The following examples show you how to decide, to choose and to solve these types of integrations.

Example 1: Determine $\int 2x\, e^{3x}\, dx$

must use integration by parts formula: $\int u\,.\,dv = u\,.\,v - \int v\,.\,du$

choose $u = 2x \rightarrow du = 2\,dx$ and $dv = e^{3x}\,dx \rightarrow v = \frac{1}{3}\,e^{3x}$

use power rule for derivative and for integration use $\int e^{kx}\,dx = \frac{1}{k}\,e^{kx} + C$

Integration by Parts gives: $\int 2x\,e^{3x}\,dx = 2x\,.\,\frac{1}{3}\,e^{3x} - \int \frac{1}{3}\,e^{3x}\,.\,2\,dx = \frac{2}{3}\,x\,e^{3x} - \int \frac{2}{3}\,e^{3x}\,dx$

$$\int 2x\,e^{3x}\,dx = \frac{2}{3}\,x\,e^{3x} - \int \frac{2}{3}\,e^{3x}\,dx = \frac{2}{3}\,x\,e^{3x} - \frac{2}{3}\,\frac{1}{3}\,e^{3x} + C = \frac{2}{3}\,x\,e^{3x} - \frac{2}{9}\,e^{3x} + C = \frac{2}{9}\,e^{3x}\,(3x-1) + C$$

<u>Example 2:</u> Determine $\int \frac{5x}{\sqrt{x+1}}\,dx$ rewrite as: $\int 5x\,(x+1)^{-1/2}\,dx$

must use integration by parts formula: $\int u\,.\,dv = u\,.\,v - \int v\,.\,du$

choose $u = 5x \rightarrow du = 5\,dx$ and $dv = (x+1)^{-1/2}\,dx \rightarrow v = 2\,(x+1)^{1/2}$

use power rule for derivative and for integration use the substitution technique:

To get v from dv : choose $u = x+1$, $du = dx \rightarrow \int u^{-1/2}\,du = 2u^{1/2} + C$

Integration by Parts gives:

$$\int 5x\,(x+1)^{-1/2}\,dx = 5x\,.\,2(x+1)^{1/2} - \int 2(x+1)^{1/2}\,.\,5\,dx = 10x(x+1)^{1/2} - \int 10(x+1)^{1/2}\,dx$$

Notice that the integral on the right side of the equation is easier to solve than the integral on the left side of the equation.

To solve $\int 10(x+1)^{1/2}\,dx$: choose same substitution: $u = x+1$, $du = dx \rightarrow \int 10u^{1/2}\,du = 10\frac{u^{3/2}}{3/2} + C$

$$\int 10u^{1/2}\,du = 10\frac{u^{3/2}}{3/2} + C = \frac{20}{3}\,u^{3/2} + C$$

$$\int 5x\,(x+1)^{-1/2}\,dx = 10x(x+1)^{1/2} - \frac{20}{3}\,(x+1)^{3/2} + C = \frac{10}{3}\,\sqrt{x+1}\,(x-2) + C$$

<u>Example 3:</u> Determine $\int x^2 e^{4x}\,dx$; must use integration by parts formula twice.

integration by parts formula: $\int u\,.\,dv = u\,.\,v - \int v\,.\,du$

use power rule for derivative and for integration use $\int e^{kx}\,dx = \frac{1}{k}\,e^{kx} + C$

choose $u = x^2 \rightarrow du = 2x\,dx$ and $dv = e^{4x}\,dx \rightarrow v = \frac{1}{4}\,e^{4x}$

Integration by Parts gives:

$$\int x^2\,e^{4x}\,dx = x^2\,.\,\tfrac{1}{4}\,e^{4x} - \int \tfrac{1}{4}\,e^{4x}\,.\,2x\,dx = \tfrac{1}{4}\,x^2\,e^{4x} - \int \tfrac{1}{2}\,x\,e^{4x}\,dx$$

Use again the integration by parts for $\int \frac{1}{2}\,x\,e^{4x}\,dx$

choose $u = \frac{1}{2}\,x \rightarrow du = \frac{1}{2}\,dx$ and $dv = e^{4x}\,dx \rightarrow v = \frac{1}{4}\,e^{4x}$

integration by parts gives us: $\int \frac{1}{2}\,x\,e^{4x}\,dx = \frac{1}{2}\,x\,\frac{1}{4}\,e^{4x} - \int \frac{1}{4}\,e^{4x}\,.\,\frac{1}{2}\,dx = \frac{1}{8}\,x\,e^{4x} - \int \frac{1}{8}\,e^{4x}\,dx$

$$\int x^2\,e^{4x}\,dx = \tfrac{1}{4}\,x^2\,e^{4x} - \left[\tfrac{1}{8}\,x\,e^{4x} - \int \tfrac{1}{8}\,e^{4x}\,dx\right] = \tfrac{1}{4}\,x^2\,e^{4x} - \tfrac{1}{8}\,x\,e^{4x} + \tfrac{1}{8}\,\tfrac{1}{4}\,e^{4x} + C$$

$$\int x^2\,e^{4x}\,dx = \tfrac{1}{4}\,x^2\,e^{4x} - \left[\tfrac{1}{8}\,x\,e^{4x} - \int \tfrac{1}{8}\,e^{4x}\,dx\right] = \tfrac{1}{4}\,x^2\,e^{4x} - \tfrac{1}{8}\,x\,e^{4x} + \tfrac{1}{32}\,e^{4x} + C$$

$$\int x^2\,e^{4x}\,dx = \tfrac{1}{32}\,e^{4x}\,(8x^2 - 4x + 1) + C$$

Tabular form (use for repeated integration by parts) for same Example 3:

The table ends when the derivatives left column has 0 . The right column represents antiderivatives. Also every row has a sign that alternates.

	u	dv
$+$	x^2	e^{4x}
$-$	$2x$	$\frac{1}{4}\,e^{4x}$
$+$	2	$\frac{1}{16}\,e^{4x}$
	0	$\frac{1}{64}\,e^{4x}$

$$\int x^2\,e^{4x}\,dx = +x^2\,\frac{e^{4x}}{4} - 2x\,\frac{e^{4x}}{16} + 2\,\frac{e^{4x}}{64} + C$$

$$\int x^2\,e^{4x}\,dx = \frac{1}{4}\,x^2\,e^{4x} - \frac{1}{8}\,x\,e^{4x} + \frac{1}{32}\,e^{4x} + C$$

$$\int x^2\,e^{4x}\,dx = \frac{1}{32}\,e^{4x}\,(8x^2 - 4x + 1) + C$$

Example 4: Determine $\int (3x+5)\, e^{2x}\, dx$

use power rule for derivative and for integration use $\int e^{kx}\, dx = \frac{1}{k}\, e^{kx} + C$

dv always has dx in the choice.

integration by parts formula: $\int u\,.\,dv = u\,.\,v - \int v\,.\,du$

choose $u = 3x+5 \rightarrow du = 3\,dx$ and $dv = e^{2x}\,dx \rightarrow v = \frac{1}{2}\,e^{2x}$

Integration by Parts gives: $\int (3x+5)\,e^{2x}\,dx = (3x+5)\,.\,\frac{1}{2}\,e^{2x} - \int \frac{1}{2}\,e^{2x}\,.\,3\,dx = \frac{1}{2}\,(3x+5)\,e^{2x} - \int \frac{3}{2}\,e^{2x}\,dx$

$$\int (3x+5)\,e^{2x}\,dx = \frac{1}{2}\,(3x+5)\,e^{2x} - \int \frac{3}{2}\,e^{2x}\,dx = \frac{1}{2}\,(3x+5)\,e^{2x} - \frac{3}{2}\,\frac{1}{2}\,e^{2x} + C$$

$$= \frac{1}{2}\,(3x+5)\,e^{2x} - \frac{3}{4}\,e^{2x} + C = \frac{1}{4}\,e^{2x}\,(6x+7) + C$$

Example 5: Determine $\int (5x+1)\,(4x+3)^4\,dx$

use power rule for derivative and for integration use the substitution technique:

integration by parts formula: $\int u\,.\,dv = u\,.\,v - \int v\,.\,du$

choose $u = 5x+1 \rightarrow du = 5\,dx$ and $dv = (4x+3)^4\,dx \rightarrow v = \frac{1}{20}\,(4x+3)^5$

To get v from dv : choose $u = 4x+3$, $du = 4\,dx$, $\frac{1}{4}\,du = dx \rightarrow \int \frac{1}{4}\,u^4\,du = \frac{1}{4}\,\frac{1}{5}\,u^5 + C = \frac{1}{20}\,u^5 + C$

Integration by Parts gives: $\int (5x+1)\,(4x+3)^4\,dx = (5x+1)\,.\,\frac{1}{20}\,(4x+3)^5 - \int \frac{1}{20}\,(4x+3)^5\,.\,5\,dx$

To solve $\int \frac{1}{20}\,(4x+3)^5\,.\,5\,dx = \int \frac{5}{20}\,(4x+3)^5\,dx$: choose same substitution:

$u = 4x+3$, $du = 4\,dx$, $\frac{1}{4}\,du = dx \rightarrow \int \frac{5}{80}\,u^5\,du = \dfrac{5}{80}\,\dfrac{u^6}{6} + C = \frac{5}{480}\,u^6 + C$

$$\int (5x+1)\,(4x+3)^4\,dx = \frac{1}{20}\,(5x+1)\,(4x+3)^5 - \frac{5}{480}\,(4x+3)^6 + C$$

$$= \frac{1}{480}\,(4x+3)^5\,(100x+9) + C$$

Example 6: Determine $\int x^3\,(4+x^2)^3\,dx = \int x^2 \,.\, x(4+x^2)^3\,dx$

use power rule for derivative and for integration use the substitution technique:

integration by parts formula: $\int u\,.\,dv = u\,.\,v - \int v\,.\,du$

choose $u = x^2 \rightarrow du = 2x\,dx$ and $dv = x(4+x^2)^3\,dx \rightarrow v = \frac{1}{8}\,(4+x^2)^4$

To get v from dv : choose $u = 4+x^2$, $du = 2x\,dx$, $\frac{1}{2}\,du = x\,dx \rightarrow \int \frac{1}{2}\,u^3\,du = \frac{1}{2}\,\frac{1}{4}\,u^4 + C = \frac{1}{8}\,u^4 + C$

Integration by Parts gives: $\int x^3\,(4+x^2)^3\,dx = x^2\,.\,\frac{1}{8}\,(4+x^2)^4 - \int \frac{1}{8}\,(4+x^2)^4\,.\,2x\,dx$

To solve $\int \frac{1}{8}\,(4+x^2)^4\,.\,2x\,dx = \int \frac{1}{4}\,(4+x^2)^4\,dx$: choose same substitution:

$u = 4+x^2$, $du = 2x\,dx$, $\frac{1}{2}\,du = x\,dx \rightarrow \int \frac{1}{8}\,u^4\,du = \dfrac{1}{8}\,\dfrac{u^5}{5} + C = \frac{1}{40}\,u^5 + C$

$$\int x^3\,(4+x^2)^3\,dx = \frac{1}{8}\,x^2\,(4+x^2)^4 - \frac{1}{40}\,(4+x^2)^5 + C = \frac{1}{10}\,(4+x^2)^4\,(x^2-1) + C$$

Example 7: Determine $\int (2x+1)^3\,e^{-x}\,dx$; must use integration by parts formula three times.

Tabular form (use for repeated integration by parts):

The table ends when the derivatives left column has 0 . The right column represents antiderivatives.
Also every row has a sign that alternates.

	u	dv
$+$	$(2x+1)^3$	e^{-x}
$-$	$6(2x+1)^2$	$-e^{-x}$
$+$	$24(2x+1)$	e^{-x}
$-$	48	$-e^{-x}$
	0	e^{-x}

$$\int (2x+1)^3\,e^{-x}\,dx$$

$$= +(2x+1)^3\,\frac{e^{-x}}{-1} - 6(2x+1)^2\,e^{-x} + 24(2x+1)\,\frac{e^{-x}}{-1} - 48\,e^{-x} + C$$

$$= -(2x+1)^3\,e^{-x} - 6(2x+1)^2\,e^{-x} - 24(2x+1)\,e^{-x} - 48\,e^{-x} + C$$

$$= -e^{-x}\,(8x^3 + 36x^2 + 78x + 79) + C$$