Definition of a Sequence

A sequence is a function represented by $\{a_n\}$

where the domain is the variable n with positive integers.

The function values: $a_1, a_2, a_3, \ldots, a_n, \ldots$ are the terms of the sequence.

A finite sequence has a defined number of terms.

Ex: $2, 6, 10, 14$ this sequence is finite with 4 terms.

An infinite sequence has an undefined number of terms.

Ex: $2, 6, 10, 14, \ldots$ this sequence is infinite.

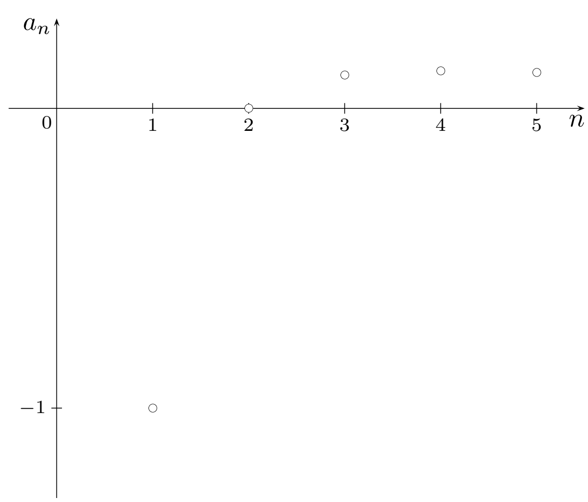
Often a pattern exists in a given sequence that will help to determine an expression called the n^{th} term, also called the general term of the sequence. In the above example, the n^{th} term is written as $a_n = 4n - 2$.

Once you have a_n ;verify as follows:

first term: $a_1 = 4(1) - 2 = 2$; second term: $a_2 = 4(2) - 2 = 6$
third term: $a_3 = 4(3) - 2 = 10$; fourth term: $a_4 = 4(4) - 2 = 14$

Example 1: Given the n^{th} term of a sequence, write the first 5 terms and show the result graphically:

$$a_n = \frac{n-2}{n^2} \longrightarrow a_1 = -1, a_2 = 0, a_3 = \tfrac{1}{9}, a_4 = \tfrac{1}{8}, a_5 = \tfrac{3}{25}$$



the first 5 terms are: $\left\{-1, 0, \tfrac{1}{9}, \tfrac{1}{8}, \tfrac{3}{25}, \ldots\right\}$

in decimals: $\left\{-1, 0, 0.1111, 0.125, 0.12\right\}$

The points (n, a_n) shown on the graph are the coordinates of these 5 terms:

$(1, -1)$; $(2, 0)$; $(3, 0.1111)$; $(4, 0.125)$; $(5, 0.12)$

Example 2: Given the n^{th} term of a sequence, write the first 5 terms and show the result graphically:

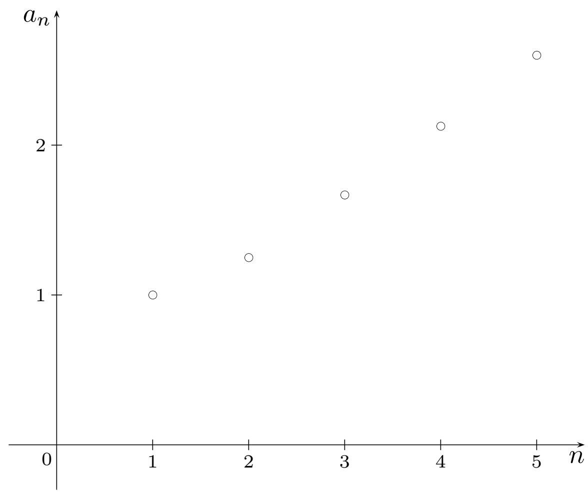
$a_n = \dfrac{n^2+1}{2n} \longrightarrow a_1 = 1\,,\ a_2 = \frac{5}{4}\,,\ a_3 = \frac{10}{6}\,,\ a_4 = \frac{17}{8}\,,\ a_5 = \frac{26}{10}$

the first 5 terms are: $\left\{1\,,\ \frac{5}{4}\,,\ \frac{5}{3}\,,\ \frac{17}{8}\,,\ \frac{13}{5}\,,\ \ldots\right\}$

in decimals: $\{1\,,\ 1.25\,,\ 1.6667\,,\ 2.125\,,\ 2.6\}$

The points (n, a_n) shown on the graph are the coordinates of these 5 terms:

$(1,1)$; $(2,1.25)$; $(3,1.6667)$; $(4,2.125)$; $(5,2.6)$



Example 3: Given the n^{th} term of a sequence, write the first 5 terms and show the result graphically:

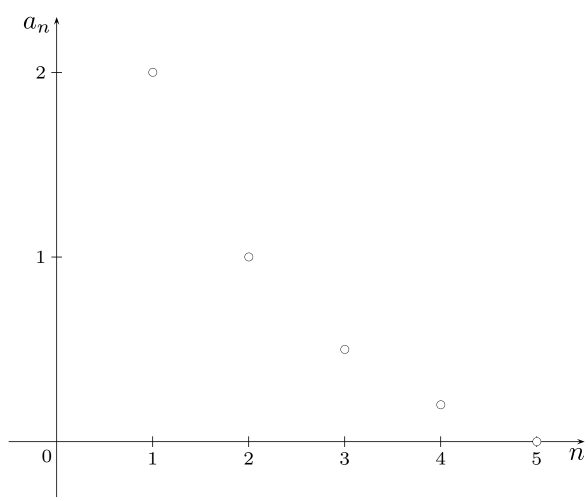
$a_n = \dfrac{5-n}{n+1} \longrightarrow a_1 = 2\,,\ a_2 = 1\,,\ a_3 = \frac{2}{4}\,,\ a_4 = \frac{1}{5}\,,\ a_5 = 0$

the first 5 terms are: $\left\{2\,,\ 1\,,\ \frac{1}{2}\,,\ \frac{1}{5}\,,\ 0\,,\ \ldots\right\}$

in decimals: $\{2\,,\ 1\,,\ 0.5\,,\ 0.2\,,\ 0\}$

The points (n, a_n) shown on the graph are the coordinates of these 5 terms:

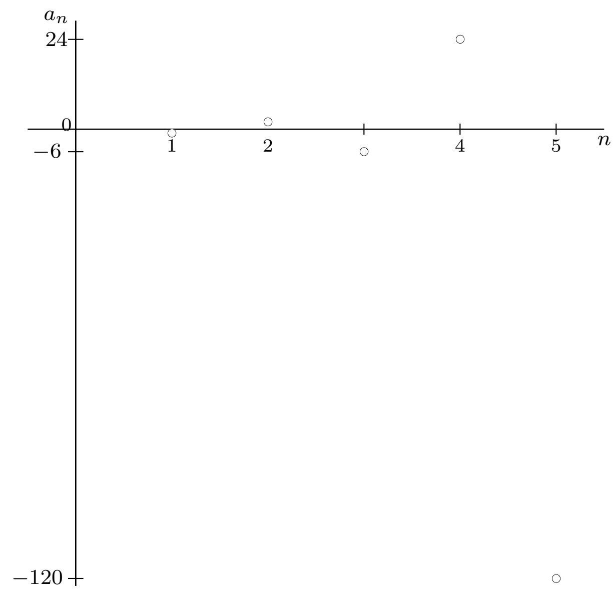
$(1,2)$; $(2,1)$; $(3,0.5)$; $(4,0.2)$; $(5,0)$



Example 4: Given the n^{th} term of a sequence, write the first 5 terms and show the result graphically:

$a_n = (-1)^n \,.\, n! \quad \longrightarrow n!$ reads n factorial $\big[$ abbreviation of $1.2.3.4...(n-1).n$ $\big]$

$a_1 = (-1)\,.\,1! = -1\,,\; a_2 = (1)\,.\,2! = 2\,,\; a_3 = (-1)\,.\,3! = -6\,,\; a_4 = (1)\,.\,4! = 24\,,\; a_5 = (-1)\,.\,5! = -120$



the first 5 terms are:
$\{-1\,,\, 2\,,\, -6\,,\, 24\,,\, -120\,,\, \ldots\}$

The points (n, a_n) shown on the graph are the coordinates of these 5 terms:

$(1,-1)$; $(2,2)$; $(3,-6)$; $(4,24)$; $(5,-120)$

Example 5: Given the n^{th} term of a sequence, write the first 5 terms.

Let $f(x) = e^{\,x/4}$ and $a_n = f^{(n-1)}(0)$ $\big[$ $f^{(5)}(0)$ means the fifth derivative of $f(x)$ at $x = 0$ $\big]$

$f^{(0)}(x) = f(x) \rightarrow a_1 = f(0) = e^0 = 1$

$f^{(1)}(x) = f'(x) \rightarrow a_2 = f'(0) = \frac{1}{4}\, e^0 = \frac{1}{4}$

$f^{(2)}(x) = f''(x) \rightarrow a_3 = f''(0) = \frac{1}{16}\, e^0 = \frac{1}{16}$

$f^{(3)}(x) \rightarrow a_4 = f^{(3)}(0) = \frac{1}{64}\, e^0 = \frac{1}{64}$

$f^{(4)}(x) \rightarrow a_5 = f^{(4)}(0) = \frac{1}{256}\, e^0 = \frac{1}{256}$

the first 5 terms are: $\left\{1\,,\, \dfrac{1}{4}\,,\, \dfrac{1}{16}\,,\, \dfrac{1}{64}\,,\, \dfrac{1}{256}\,,\, \ldots\right\}$

Example 6: Given the n^{th} term of a sequence, write the first 5 terms.

Let $f(x) = \sin(3x)$ and $a_n = f^{(n-1)}(0)$ $\left[\ f^{(7)}(0)\right.$ means the seventh derivative of $f(x)$ at $\left. x = 0\ \right]$

$f^{(0)}(x) = f(x) \rightarrow a_1 = f(0) = \sin(0) = 0$

$f^{(1)}(x) = f'(x) \rightarrow a_2 = f'(0) = 3\,\cos(0) = 3$

$f^{(2)}(x) = f''(x) \rightarrow a_3 = f''(0) = -9\,\sin(0) = 0$

$f^{(3)}(x) \rightarrow a_4 = f^{(3)}(0) = -27\,\cos(0) = -27$

$f^{(4)}(x) \rightarrow a_5 = f^{(4)}(0) = 81\,\sin(0) = 0$

the first 5 terms are: $\{\,0\,,\,3\,,\,0\,,\,-27\,,\,0\,,\,\ldots\}$

General Term of a Sequence

Given the terms of 5 sequences, find the general term of each sequence:

Look for a pattern of the terms in each sequence, then verify the answers.

(1) $\dfrac{1}{3}\,,\,\dfrac{3}{6}\,,\,\dfrac{5}{9}\,,\,\dfrac{7}{12}\,,\,\dfrac{9}{15}\,,\,\ldots$ $\qquad a_n = \dfrac{2n-1}{3n}$

(2) $\dfrac{5}{1}\,,\,\dfrac{7}{4}\,,\,\dfrac{9}{9}\,,\,\dfrac{11}{16}\,,\,\dfrac{13}{25}\,,\,\ldots$ $\qquad a_n = \dfrac{2n+3}{n^2}$

(3) $\dfrac{2}{1}\,,\,\dfrac{3}{3}\,,\,\dfrac{4}{5}\,,\,\dfrac{5}{7}\,,\,\dfrac{6}{9}\,,\,\ldots$ $\qquad a_n = \dfrac{n+1}{2n-1}$

(4) $-1\,,\,\dfrac{1}{2}\,,\,-\dfrac{1}{3}\,,\,\dfrac{1}{4}\,,\,-\dfrac{1}{5}\,,\,\ldots$ $\qquad a_n = \dfrac{\cos(\pi\,n)}{n}$

(5) $\dfrac{e}{2}\,,\,\dfrac{e^2}{3}\,,\,\dfrac{e^3}{4}\,,\,\dfrac{e^4}{5}\,,\,\dfrac{e^5}{6}\,,\,\ldots$ $\qquad a_n = \dfrac{e^n}{n+1}$

Some general term may have more than one answer.

Application of Sequence (1)

\$5000 is invested at 3% compounded monthly.

Use the n^{th} term $a_n = P\left[1+\dfrac{i}{m}\right]^n$ where P is the amount invested, i is the interest in decimal,

m number of compound intervals per year t time in years $n = m\,.\,t$ total number of compound intervals.

Find a sequence of the monthly balances for a period of one year.

replacing P with 5000 , i with 0.03 , m with 12 ; we get: $a_n = 5000\left(1+\dfrac{0.03}{12}\right)^n = 5000(1.0025)^n$

we write the first 12 terms:

$a_1 = 5000(1.0025)^1 = 5012.50$; $a_2 = 5000(1.0025)^2 = 5025.03$; $a_3 = 5000(1.0025)^3 = 5037.59$

$a_4 = 5000(1.0025)^4 = 5050.19$; $a_5 = 5000(1.0025)^5 = 5062.81$; $a_6 = 5000(1.0025)^6 = 5075.47$

$a_7 = 5000(1.0025)^7 = 5088.16$; $a_8 = 5000(1.0025)^8 = 5100.88$; $a_9 = 5000(1.0025)^9 = 5113.63$

$a_{10} = 5000(1.0025)^{10} = 5126.42$; $a_{11} = 5000(1.0025)^{11} = 5139.23$; $a_{12} = 5000(1.0025)^{12} = 5152.08$

Definition of an Arithmetic Sequence

An Arithmetic Sequence is a sequence such that the difference d between two consecutive terms is constant.

Therefore $a_2 - a_1 = d$; $a_7 - a_6 = d$ and so on.

The n^{th} term of an arithmetic sequence is $\boxed{a_n = a_1 + (n-1)d}$

Example 7: Given the arithmetic sequence: $6\,,\,4\,,\,2\,,\,0\,,\,-2\,,\,\ldots$

find d , a_n , a_{50} and which term equals -40?

difference: $d = a_2 - a_1 = 4 - 6 = -2$ same as $a_5 - a_4 = -2 - 0 = -2$ therefore $d = -2$

$a_n = a_1 + (n-1)d = 6 + (n-1)(-2) = 8 - 2n$ therefore $a_n = 8 - 2n$

$a_{50} = 8 - 2(50) = -92$ the fiftieth term equals to -92

$a_n = 8 - 2n = -40 \rightarrow 8 + 40 = 2n \rightarrow n = 24$

The twenty-fourth term equals to -40

Example 8: Given the arithmetic sequence: $11\,,\,15\,,\,19\,,\,23\,,\,27\,,\,\ldots$

find d , a_n , a_{75} and which term equals 91?

difference: $d = a_2 - a_1 = 15 - 11 = 4$ same as $a_5 - a_4 = 27 - 23 = 4$ therefore $d = 4$

$a_n = a_1 + (n-1)d = 11 + (n-1)(4) = 4n + 7$ therefore $a_n = 4n + 7$

$a_{75} = 4(75) + 7 = 307$ the seventy-fifth term equals to 307

$a_n = 4n + 7 = 91 \rightarrow 4n = 91 - 7 \rightarrow n = 21$

The twenty-first term equals to 91

Example 9: Given the arithmetic sequence: $\frac{1}{3}\,,\,\frac{5}{6}\,,\,\frac{4}{3}\,,\,\frac{11}{6}\,,\,\frac{7}{3}\,,\,\ldots$

find d , a_n , a_{30} and which term equals $\dfrac{19}{3}$?

difference: $d = a_2 - a_1 = \frac{5}{6} - \frac{1}{3} = \frac{1}{2}$ same as $a_5 - a_4 = \frac{7}{3} - \frac{11}{6} = \frac{1}{2}$ therefore $d = \frac{1}{2}$

$a_n = a_1 + (n-1)d = \frac{1}{3} + (n-1)\left(\frac{1}{2}\right) = \frac{1}{2}\,n - \frac{1}{6}$ therefore $a_n = \dfrac{3n-1}{6}$

$a_{30} = \dfrac{3(30)-1}{6} = \dfrac{89}{6}$ the thirtieth term equals to $\dfrac{89}{6}$

$a_n = \dfrac{3n-1}{6} = \dfrac{19}{3} \rightarrow 3n - 1 = 38 \rightarrow n = 13$

The thirteenth term equals to $\dfrac{19}{3}$

Example 10: Given the arithmetic sequence: $\frac{2}{5}\,,\,\frac{1}{5}\,,\,0\,,\,-\frac{1}{5}\,,\,-\frac{2}{5}\,,\,\ldots$

find d , a_n , a_{43} and which term equals -9?

difference: $d = a_2 - a_1 = \frac{1}{5} - \frac{2}{5} = -\frac{1}{5}$ same as $a_5 - a_4 = -\frac{2}{5} - \left(-\frac{1}{5}\right) = -\frac{1}{5}$ therefore $d = -\frac{1}{5}$

$a_n = a_1 + (n-1)d = \frac{2}{5} + (n-1)\left(-\frac{1}{5}\right) = -\frac{1}{5}\,n + \frac{3}{5}$ therefore $a_n = \dfrac{3-n}{5}$

$a_{43} = \dfrac{3-43}{5} = -8$ the forty-third term equals to -8

$a_n = \dfrac{3-n}{5} = -9 \rightarrow 3 - n = -45 \rightarrow n = 48$

The forty-eighth term equals to -9

Example 11: Given the arithmetic sequence: $5\,,\,20\,,\,35\,,\,50\,,\,65\,,\,\ldots$

find d , a_n , a_{55} and which term equals 230?

difference: $d = a_2 - a_1 = 20 - 5 = 15$ same as $a_5 - a_4 = 65 - 50 = 15$ therefore $d = 15$

$a_n = a_1 + (n-1)d = 5 + (n-1)(15) = 15n - 10$ therefore $a_n = 15n - 10$

$a_{55} = 15(55) - 10 = 815$ the fifty-fifth term equals to 815

$a_n = 15n - 10 = 230 \rightarrow 15n = 240 \rightarrow n = 16$

The sixteenth term equals to 230

Definition of a Geometric Sequence

A Geometric Sequence is a sequence such that the ratio r among two consecutive terms is constant.

Therefore $\dfrac{a_2}{a_1} = r$; $\dfrac{a_9}{a_8} = r$ and so on.

The n^{th} term of a geometric sequence is $\boxed{a_n = a_1\,.\,(r)^{n-1}}$

Example 12: Given the geometric sequence: $120\,,\,60\,,\,30\,,\,15\,,\,\frac{15}{2}\,,\,\ldots$

find r , a_n , a_{10} and which term equals $\dfrac{15}{32}$?

ratio: $r = \dfrac{a_2}{a_1} = \dfrac{60}{120} = \dfrac{1}{2}$ same as $\dfrac{a_5}{a_4} = \dfrac{\frac{15}{2}}{15} = \dfrac{1}{2}$ therefore $r = \dfrac{1}{2}$

$a_n = a_1\,.\,(r)^{n-1} = 120\,.\,\left(\dfrac{1}{2}\right)^{n-1}$ therefore $a_n = 120\left(\dfrac{1}{2}\right)^{n-1}$

$a_{10} = 120\left(\dfrac{1}{2}\right)^{10-1} = \dfrac{15}{64}$ the tenth term equals to $\dfrac{15}{64}$

$a_n = 120\left(\dfrac{1}{2}\right)^{n-1} = \dfrac{15}{32} \rightarrow \left(\dfrac{1}{2}\right)^{n-1} = \dfrac{1}{256} = \left(\dfrac{1}{2}\right)^{9-1} \rightarrow n = 9$

The ninth term equals to $\dfrac{15}{32}$

Example 13: Given the geometric sequence: $50\,,\,150\,,\,450\,,\,1350\,,\,4050\,,\,\ldots$ find $r\,,\,a_n\,,\,a_9\,.$

ratio: $r=\dfrac{a_2}{a_1}=\dfrac{150}{50}=3$ same as $\dfrac{a_5}{a_4}=\dfrac{4050}{1350}=3$ therefore $r=3$

$a_n=a_1\,.\,(r)^{n-1}=50\,.\,(3)^{n-1}$ therefore $a_n=50\;(3)^{n-1}$

$a_9=50\;(3)^{9-1}=328050$ the ninth term equals to 328050

Example 14: Given the geometric sequence: $-8\,,\,4\,,\,-2\,,\,1\,,\,-\frac{1}{2}\,,\,\ldots$ find $r\,,\,a_n\,,\,a_{11}\,.$

ratio: $r=\dfrac{a_2}{a_1}=\dfrac{4}{-8}=-\dfrac{1}{2}$ same as $\dfrac{a_5}{a_4}=\dfrac{-\frac{1}{2}}{1}=-\dfrac{1}{2}$ therefore $r=-\dfrac{1}{2}$

$a_n=a_1\,.\,(r)^{n-1}=-8\,.\,\left(-\dfrac{1}{2}\right)^{n-1}$ therefore $a_n=-8\left(-\dfrac{1}{2}\right)^{n-1}$

$a_{11}=-8\left(-\dfrac{1}{2}\right)^{11-1}=-\dfrac{1}{128}$ the eleventh term equals to $-\dfrac{1}{128}$

Example 15: Given the geometric sequence: $17\,,\,-34\,,\,68\,,\,-136\,,\,272\,,\,\ldots$ find $r\,,\,a_n\,,\,a_8\,.$

ratio: $r=\dfrac{a_2}{a_1}=\dfrac{-34}{17}=-2$ same as $\dfrac{a_5}{a_4}=\dfrac{272}{-136}=-2$ therefore $r=-2$

$a_n=a_1\,.\,(r)^{n-1}=17\,.\,(-2)^{n-1}$ therefore $a_n=17\;(-2)^{n-1}$

$a_8=17\;(-2)^{8-1}=-2176$ the eighth term equals to -2176

Example 16: Given the geometric sequence: $22\,,\,66\,,\,198\,,\,594\,,\,1782\,,\,\ldots$

find $r\,,\,a_n\,,\,a_{12}$ and which term equals 48114 ?

ratio: $r=\dfrac{a_2}{a_1}=\dfrac{66}{22}=3$ same as $\dfrac{a_5}{a_4}=\dfrac{1782}{594}=3$ therefore $r=3$

$a_n=a_1\,.\,(r)^{n-1}=22\,.\,(3)^{n-1}$ therefore $a_n=22\;(3)^{n-1}$

$a_{12}=22\;(3)^{12-1}=3897234$ the twelfth term equals to 3897234

$a_n=22\;(3)^{n-1}=48114\rightarrow(3)^{n-1}=2187=(3)^{8-1}\rightarrow n=8$

The eighth term equals to 48114

Identify Types of Sequence

Given 5 sequences, identify which sequence is arithmetic, geometric or neither:

Look for a pattern of the terms in each sequence, then verify the answers.

(1) $18, 16, 14, 12, 10, \ldots$ arithmetic with $d = -2$

(2) $18, 9, \frac{9}{2}, \frac{9}{4}, \frac{9}{8}, \ldots$ geometric with $r = \frac{1}{2}$

(3) $18, -18, 18, -18, 18, \ldots$ geometric with $r = -1$

(4) $18, 10, 22, 6, 30, \ldots$ neither

(5) $18, 20, 22, 24, 26, \ldots$ arithmetic with $d = 2$

Application of Sequence (2)

An employee has an initial salary of \$28000. The salary increases 3% per year.

Use the n^{th} term $a_n = P\,[1+i]^n$ where P is the initial salary, i is the rate of increase in decimal,

n is yearly term.

Find a sequence of the first 3 years salaries.

replacing P with 28000 , i with 0.03 ; we get: $a_n = 28000\,(1.03)^n$

write the first 3 terms:

$a_1 = 28000(1.03)^1 = 28840.00$; $a_2 = 28000(1.03)^2 = 29705.20$; $a_3 = 28000(1.03)^3 = 30596.36$

Limit of a Sequence

Given a sequence with the n^{th} term , find if the sequence converges or diverges.

To do it, need to evaluate $\boxed{\lim\limits_{n\to\infty} a_n}$

it is convergent if the result is a real number;

it is divergent if the result is undefined.

The limit of a sequence can also be verified graphically (not always reliable).

Note: If the result of the limit is indeterminate, may use L'Hôpital's Rule to remove the indeterminate form.

Examples of Limit of a Sequence

In Example 1: $\lim\limits_{n\to\infty} \dfrac{n-2}{n^2} = \dfrac{\infty}{\infty} \longrightarrow$ use L'Hôpital's Rule: $\lim\limits_{n\to\infty} \dfrac{1}{2n} = \dfrac{1}{\infty} = 0$

OR

$$\lim_{n\to\infty} \frac{\frac{n}{n^2} - \frac{2}{n^2}}{\frac{n^2}{n^2}} = \lim_{n\to\infty} \frac{\frac{1}{n} - \frac{2}{n^2}}{1} = \frac{0-0}{1} = 0$$

the sequence converges; verify the graph shown in Example 1.

In Example 2: $\lim\limits_{n\to\infty} \dfrac{n^2+1}{2n} = \dfrac{\infty}{\infty} \longrightarrow$ use L'Hôpital's Rule: $\lim\limits_{n\to\infty} \dfrac{2n}{2} = \lim\limits_{n\to\infty} n = \infty$

OR

$$\lim_{n\to\infty} \frac{\frac{n^2}{n^2} + \frac{1}{n^2}}{2\frac{n}{n^2}} = \lim_{n\to\infty} \frac{1 + \frac{1}{n^2}}{2\frac{1}{n}} = \frac{1-0}{0} = \text{undefined}$$

the sequence diverges; verify the graph shown in Example 2.

In Example 3: $\lim\limits_{n\to\infty} \dfrac{5-n}{n+1} = \dfrac{-\infty}{\infty} \longrightarrow$ use L'Hôpital's Rule: $\lim\limits_{n\to\infty} \dfrac{-1}{1} = -1$

OR

$$\lim_{n\to\infty} \frac{\frac{5}{n} - \frac{n}{n}}{\frac{n}{n} + \frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{5}{n} - 1}{1 + \frac{1}{n}} = \frac{0-1}{1+0} = -1$$

the sequence converges; when you verify the graph shown in Example 3 (it is not reliable).

In Example 4: $\lim\limits_{n\to\infty} (-1)^n \,.\, n! = \pm 1 \lim\limits_{n\to\infty} \left[n(n-1)(n-2)(n-3)\ldots 3.2.1 \right] = \pm\infty$; the sequence diverges.

In Example 5: the sequence: $\left\{ 1\,, \dfrac{1}{4}\,, \dfrac{1}{16}\,, \dfrac{1}{64}\,, \dfrac{1}{256}\,, \ldots \right\}$ the general term $a_n = \dfrac{1}{4^{n-1}}$

geometric sequence with $r = \dfrac{1}{4}$; $-1 < r < 1$; $\lim\limits_{n\to\infty} \dfrac{1}{4^{n-1}} = 0$; the sequence converges.

In Example 6: the sequence: $\{ 0\,, 3\,, 0\,, -27\,, 0\,, \ldots \}$
the terms show oscillating since the negative sign alternates and the absolute value increases. The sequence diverges.

Definition of a Series

A series is the sum of all terms of a sequence $\{a_n\}$

Example: Given a finite sequence: $4\,,\,7\,,\,11\,,\,16\,,\,22\,,\,31$ find the series.

The series is $4+7+11+16+22+31$ the sum of the six terms of the sequence.

The sum S of the series is 91.

Notation: A finite series is written as $\sum_{k=1}^{N} a_k$ in the above example, it is written as $\sum_{k=1}^{6} a_k$

It is called the sigma notation.

An infinite series is written as $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + \ldots$

Sequence of a Partial Sums of an Infinite Series

Given an infinite series $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + \ldots$

the partial sums of the series is defined by:

$S_1 = a_1\ ;\ S_2 = a_1 + a_2 = S_1 + a_2\ ;\ S_3 = a_1 + a_2 + a_3 = S_2 + a_3$

$S_4 = a_1 + a_2 + a_3 + a_4 = S_3 + a_4\ ;\ S_5 = a_1 + a_2 + a_3 + a_4 + a_5 = S_4 + a_5$ and so on.

An infinite sequence $\{S_n\}$ of partial sums of the series is written as $S_1\,,\,S_2\,,\,S_3\,,\,S_4\,,\,S_5\,,\,\ldots\,,\,S_n$

where $S_n = \sum_{k=1}^{n} a_k$ is the general term of the sequence.

Convergence & Divergence of an Infinite Series

Given an infinite series $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + \ldots$ and the sequence of the partial sums $\{S_n\}$,

then the infinite series converges to a real number if $\lim_{n\to\infty} S_n = \sum_{n=1}^{\infty} a_n =$ a real number (sum of the series).

If $\lim_{n\to\infty} S_n =$ does not exist, then the series diverges.

Note: This method does not always work since it is hard to find the general term S_n .

The way to proceed is to find a pattern in the partial sums and make an estimate of the sum of the series; the series is convergent.

If the estimate can't be done, then no sum exists for the series; the series is divergent.

Example 17: Given an infinite series $\sum_{n=1}^{\infty} 4^n = 4 + 16 + 64 + 256 + 1024 + \ldots$

Determine if the series is convergent or divergent.

The sequence of partial sums $\{S_n\}$ are:

$S_1 = 4\,,\ S_2 = 20\,,\ S_3 = 84\,,\ S_4 = 340\,,\ S_5 = 1364$

the pattern in the partial sums clearly shows that the values increase, then no sum exists for the series; conclusion: the series is divergent.

Example 18: Given an infinite series $\sum_{n=1}^{\infty} \frac{5}{2^n} = \frac{5}{2^1} + \frac{5}{2^2} + \frac{5}{2^3} + \frac{5}{2^4} + \frac{5}{2^5} + \ldots$

Determine if the series is convergent or divergent.

The sequence of partial sums $\{S_n\}$ are:

$$S_1 = \frac{5}{2^1} = 2.5 \ ; \ S_2 = \frac{5}{2^1} + \frac{5}{2^2} = \frac{15}{4} = 3.75 \ ; \ S_3 = \frac{5}{2^1} + \frac{5}{2^2} + \frac{5}{2^3} = \frac{35}{8} = 4.375$$

$$S_4 = \frac{5}{2^1} + \frac{5}{2^2} + \frac{5}{2^3} + \frac{5}{2^4} = \frac{75}{16} = 4.6875 \ ; \ S_5 = \frac{5}{2^1} + \frac{5}{2^2} + \frac{5}{2^3} + \frac{5}{2^4} + \frac{5}{2^5} = \frac{155}{32} = 4.84375$$

the pattern in the partial sums clearly shows that the values tend toward the number 5, then the sum exists for the series; conclusion: the series is convergent.

Properties of Infinite Properties

Given two convergent series $\sum_{n=1}^{\infty} a_n = N_a$ and $\sum_{n=1}^{\infty} b_n = N_b$ with c a real number,

then the following properties are valid:

$$\sum_{n=1}^{\infty} c\,a_n = c \sum_{n=1}^{\infty} a_n = c\,N_a$$

$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n = N_a + N_b$$

n^{th} term test for Divergence

If an infinite series $\sum_{n=1}^{\infty} a_n$ has $\lim_{n\to\infty} a_n \neq 0$ then the series is divergent.

Note: This test does not imply that if $\lim_{n\to\infty} a_n = 0$, the series is necessarily convergent.
Another test must be used to confirm convergence or divergence.

Example 19: Given an infinite series $\sum_{n=1}^{\infty} \frac{3-n}{2n+5}$; test the series for divergence.

Use the n^{th} term test for Divergence: $\lim_{n\to\infty} \frac{3-n}{2n+5} = \frac{-\infty}{\infty}$

Note: Sometimes use L'Hôpital's Rule to evaluate indeterminate forms only like $\frac{0}{0}$ or $\frac{\infty}{\infty}$

using L'Hôpital's Rule: $\lim_{n\to\infty} \frac{3-n}{2n+5} = \lim_{n\to\infty} \frac{-1}{2} = -\frac{1}{2} \neq 0$; conclusion: the series is divergent.

Example 20: Given an infinite series $\sum_{n=1}^{\infty} \frac{6}{5^n}$; test the series for divergence.

Use the n^{th} term test for Divergence: $\lim_{n\to\infty} \frac{6}{5^n} = 0$; conclusion: the test fails.

Note: Another test must be made to confirm the convergence or divergence of the series. (shown later)

Definition of an Arithmetic Series

An infinite arithmetic series is written as:

$$\sum_{n=1}^{\infty} \left[a_1 + (n-1)d\right] = a_1 + (a_1 + 1d) + (a_1 + 2d) + (a_1 + 3d) + \cdots + (a_1 + nd) + \ldots$$

where $a_n = a_1 + (n-1)d$ with difference d as defined for arithmetic sequence.

The sum of n terms of an arithmetic series: $\boxed{S_n = \frac{n}{2}\ (a_1 + a_n)}$

An infinite arithmetic series $\sum_{n=1}^{\infty} \left[a_1 + (n-1)d\right]$ always diverges

Example 21: Given an arithmetic general term $an = 3n - 7$; find the sum of the first 40 terms.

use $S_n = \frac{n}{2}\ (a_1 + a_n)$; need $a_1 = 3(1) - 7 = -4$ and $a_{40} = 3(40) - 7 = 113$

the sum of the first 40 terms: $S_{40} = \frac{40}{2}\ (-7 + 113) = 2180$

Definition of a Geometric Series

An infinite geometric series is written as: $\displaystyle\sum_{n=0}^{\infty} a\,.\,r^n = a + a\,r + a\,r^2 + a\,r^3 + \cdots + a\,r^n + \ldots$

where $a_n = a_1\,.\,r^{n-1}$ with ratio r as defined for geometric sequence.

Note: summation notation starts at $n = 0$

The N^{th} partial sum of the geometric series is $\boxed{S_N = \dfrac{a\left(1 - r^{N+1}\right)}{1 - r}}$ with $r \neq 1$

An infinite geometric series $\displaystyle\sum_{n=0}^{\infty} a\,.\,r^n$ diverges if $|r| \geq 1$ meaning $r \leq -1$ or $r \geq 1$

An infinite geometric series $\displaystyle\sum_{n=0}^{\infty} a\,.\,r^n$ converges if $|r| < 1$ meaning $-1 < r < 1$ and

the sum of $\boxed{\displaystyle\sum_{n=0}^{\infty} a\,.\,r^n = S_\infty = \frac{a}{1-r}}$

Example 22: Find the fourth and seventh partial sum of the geometric series $\displaystyle\sum_{n=0}^{\infty} \frac{7}{2^n}$

Use first term $(n = 0)$ $a = 7$, $r = \frac{1}{2}$ in the formula: $S_N = \dfrac{a\left(1 - r^{N+1}\right)}{1 - r}$

Fourth partial sum: $S_4 = \dfrac{7\left(1 - \left(\frac{1}{2}\right)^5\right)}{1 - \frac{1}{2}} = \dfrac{217}{16} \approx 13.6$

Seventh partial sum: $S_7 = \dfrac{7\left(1 - \left(\frac{1}{2}\right)^8\right)}{1 - \frac{1}{2}} = \dfrac{1785}{128} \approx 13.9453$

Example 23: In Example 20, the infinite series $\displaystyle\sum_{n=1}^{\infty} \frac{6}{5^n}$; failed the n^{th} term test for Divergence

Notice that the series is geometric with $\displaystyle\sum_{n=0}^{\infty} \frac{6}{5^{n+1}}$ first term $a = \frac{6}{5}$ and ratio $r = \frac{1}{5}$

since $-1 < r < 1$; series is convergent and sum is $S_\infty = \dfrac{a}{1-r} = \dfrac{\frac{6}{5}}{1 - \frac{1}{5}} = \dfrac{3}{2}$

Example 24: Verify if the geometric series $\sum_{n=0}^{\infty} 2\,(0.3)^n$ converges, if it does, find the sum.

Use first term $(n=0)$ $a=2$, $r=0.3 \rightarrow |r|<1$ then series converges and

the formula for the sum is $S_\infty = \dfrac{a}{1-r} = \dfrac{2}{1-0.3} = \dfrac{20}{7} \approx 2.86$

Example 25: Verify if the geometric series $\sum_{n=0}^{\infty} 2\left(\frac{5}{3}\right)^n$ converges, if it does, find the sum.

Use first term $(n=0)$ $a=2$, $r=\dfrac{5}{3} \rightarrow |r|>1$ then the geometric series diverges and no sum exists.

Example 26:

Given the repeated decimal $0.\overline{8}$, express it as a geometric series.
Find the sum and write the decimal as the ratio of two integers.

Write $0.\overline{8} = 0.8 + 0.08 + 0.008 + \cdots = \sum_{n=0}^{\infty} 0.8\,(0.1)^n$

Use first term $(n=0)$ $a=0.8$, $r=0.1 \rightarrow |r|<1$ then series converges and

the formula for the sum is $S_\infty = \dfrac{a}{1-r} = \dfrac{0.8}{1-0.1} = \dfrac{8}{9}$

Example 27:

Given the repeated decimal $2.\overline{01}$, express it as a geometric series.
Find the sum and write the decimal as the ratio of two integers.

Write $2.\overline{01} = 2 + 0.01 + 0.0001 + 0.000001 + \cdots = 2 + \sum_{n=0}^{\infty} 0.01\,(0.01)^n = 2 + \sum_{n=0}^{\infty} (0.01)^{n+1}$

Use first term $(n=0)$ $a=0.01$, $r=0.01 \rightarrow |r|<1$ then series converges and

the formula for the sum is $S_\infty = \dfrac{a}{1-r}$; therefore the sum is $2 + \dfrac{0.01}{1-0.01} = 2 + \dfrac{1}{99} = \dfrac{199}{99}$

Example 28: Given a geometric general term $a_n = \dfrac{3}{2^n}$; find the sum of the first 10 terms.

use $\sum_{n=0}^{9} \dfrac{3}{2^n} \rightarrow$ need first term $(n=0)$ $a=3$, $r=\frac{1}{2}$ in the formula: $S_N = \dfrac{a\left(1-r^{N+1}\right)}{1-r}$

the sum of the first 10 terms: $S_9 = \dfrac{3\left(1-\left(\frac{1}{2}\right)^{10}\right)}{1-\frac{1}{2}} = \dfrac{3069}{512} \approx 2.997$

Annuity

A deposit P in dollars is made m times a year for a number of years that pays $i\%$ compounded m times a year.

Find the balance in the account at the end of t number of years.

Use the formula $\boxed{A_n = P\left(1+\dfrac{i}{m}\right)^n}$

where P is the amount deposited in dollars.

t is the number of years.

i is annual interest rate in decimal form.

m is the number of times per year.

$n = m\,.\,t$ is the total number of times for the duration.

A_n is the general term of the sequence.

Example 29:

A deposit of \$25 is made monthly for 4 years that pays 3% compounded monthly.
Find the balance in the account at the end of 4 years.

Use $P = 25$, $m = 12$, $i = 0.03$, $t = 4$, $n = (12)(4) = 48$ and we get the sum: $\displaystyle\sum_{k=1}^{n} P\left(1+\frac{i}{m}\right)^k$

$$S_{48} = \sum_{k=1}^{48} 25\left(1+\frac{0.03}{12}\right)^k = \sum_{k=1}^{48} 25\,(1.0025)^k$$

The N^{th} partial sum formula starts at $n = 0$ so do the following adjustments:

$$S_{48} = \sum_{k=1}^{48} 25\,(1.0025)^k = -25 + \sum_{k=0}^{48} 25\,(1.0025)^k \text{ with ratio } r = 1.0025$$

Use the N^{th} partial sum formula of geometric series: $S_N = \dfrac{a\left(1-r^{N+1}\right)}{1-r}$

$$S_{48} = -25 + \frac{25\left(1-(1.0025)^{49}\right)}{1-1.0025} \approx \$1276.46$$

Applications of Sequences and Series: Annuities

1. A deposit of 40 dollars is made at the beginning of each week for 3 years into an account that pays 1.05% interest, compounded weekly a year.
   Find the balance in the account at the end of the 3 years.
   $-40+\sum_{n=0}^{156} 40\left(1+\frac{0.0105}{52}\right)^n \approx \$6339.95\ ;\ r=1.0020193861$

2. A deposit of \$300 is made that earns 3% interest, compounded monthly.
   Find a sequence that represents the monthly balances.
   Write the general term and find the first 5 terms of the sequence.
   $A_n=P\left(1+\frac{i}{12}\right)^n \implies A_1=300\left(1+\frac{0.03}{12}\right)^1=300\,(1.0025)^1=300.75$
   $A_2=301.50\ ;\ A_3=302.26\ ;\ A_4=303.01\ ;\ A_5=303.77$

3. A deposit of \$300 is made that earns 3% interest, compounded bi-weekly.
   Find a sequence that represents the bi-weekly balances.
   Write the general term and find the first 5 terms of the sequence.
   $A_n=P\left(1+\frac{i}{26}\right)^n \implies A_1=300\left(1+\frac{0.03}{26}\right)^1=300\,(1.00115384)=300.35$
   $A_2=300.69\ ;\ A_3=301.04\ ;\ A_4=301.39\ ;\ A_5=301.73$

4. To create a scholarship of \$1000 to be awarded every year, use the series $\sum 1000\,e^{-0.08n}$ to determine the sum of money required to deposit at a rate of 8% interest per year compounded continuously.
   Find the sum.
   G.S.: $a_1=1000\,e^{-0.08}\approx 923.1163\ ;\ r=e^{-0.08}\approx 0.9231163 \implies \sum_{n=1}^{\infty} 1000\,e^{-0.08n}\approx \$12006.67$

5. A patient is given 7 units of a drug daily for a long period of time. A portion of the drug remains in the patient's body after n years is given by $\sum 7\,e^{-n/3}$. Find the total amount of drug remaining in the patient's body after 5 years, 10 years and many years later.
   G.S.: $a_1=7\,e^{-1/3}\approx 5.015719468\ ;\ r=e^{-1/3}\approx 0.71653131$
   $$S_5=\frac{7\,e^{-1/3}\left[1-(e^{-1/3})^5\right]}{1-e^{-1/3}}\approx 14.35\ ;\ S_{10}\approx 17.06\ ;\ S_\infty=\frac{a_1}{1-r}=\frac{7\,e^{-1/3}}{1-e^{-1/3}}\approx 17.69$$