Substitution for the General Power Rule & Composite expressions

The integral is given in the form of an expression to a power n a real number and multiply to an expression representing a constant multiple of the derivative of the expression inside the power n.
The integral is called the chain rule integration since the given derivative is obtained using the general power rule or chain rule formula.
The chain rule is applied to a composite function like u^n or e^u or $\sin(u)$ or $\ln(u)$.

Use the substitution procedure if the description outlined above is satisfied.

Procedure: (General Power Rule)

Change the variable x into a variable u

Let $u =$ expression inside the power n

$du =$ (derivative of the expression inside the power n) dx

make appropriate adjustments to du and replace the original integral with x to a new integral with u.

The purpose is to get an easier integral to solve.

Once the answer is obtained with u, get the answer with x.

Example 1: Determine $\int 4\,(6x+5)^7\,dx$

Let $u = 6x+5$ (expression inside power $n = 7$)

$du = 6\,dx$ (derivative of the inside expression)

the adjustment is necessary to replace into the original integral; we get: $dx = \frac{1}{6}\,du$

replace $(6x+5)^7$ with u^7 and dx with $\frac{1}{6}\,du$; we get:

$$\int 4\,(6x+5)^7\,dx \longrightarrow \int 4\,u^7\,\frac{1}{6}\,du = \int \frac{2}{3}\,u^7\,du = \frac{2}{3}\,\frac{u^8}{8} + C = \frac{1}{12}\,u^8 + C \quad \text{(power rule is used)}$$

Therefore, replace $u = 6x+5$ to get answer: $\int 4\,(6x+5)^7\,dx = \frac{1}{12}\,(6x+5)^8 + C$

Example 2: Determine $\int 2x\sqrt{4+x^2}\,dx$

Let $u = 4 + x^2$ (expression inside power $n = \frac{1}{2}$)

$du = 2x\,dx$ (derivative of the inside expression)

no adjustment is necessary to replace into the original integral

replace $(4+x^2)^{1/2}$ with $u^{1/2}$ and $2x\,dx$ with du ; we get:

$$\int 2x\,(4+x^2)^{1/2}\,dx \longrightarrow \int u^{1/2}\,du = \frac{u^{3/2}}{3/2} + C = \frac{2}{3}\,u^{3/2} + C \text{ (power rule is used)}$$

Therefore, replace $u = 4 + x^2$ to get answer: $\int 2x\,(4+x^2)^{1/2}\,dx = \frac{2}{3}\,(4+x^2)^{3/2} + C$

Example 3: Determine $\int \frac{2x-1}{\sqrt{3-x+x^2}}\,dx$ rewrite: $\int (2x-1)(3-x+x^2)^{-1/2}\,dx$

Let $u = 3 - x + x^2$ (expression inside power $n = -\frac{1}{2}$)

$du = (-1+2x)\,dx$ (derivative of the inside expression)

no adjustment is necessary to replace into the original integral

replace $(3-x+x^2)^{-1/2}$ with $u^{-1/2}$ and $(2x-1)\,dx$ with du ; we get:

$$\int (2x-1)\,(3-x+x^2)^{-1/2}\,dx \longrightarrow \int u^{-1/2}\,du = \frac{u^{1/2}}{1/2} + C = 2\,u^{1/2} + C \text{ (power rule is used)}$$

Therefore, replace $u = 3 - x + x^2$ to get answer:

$$\int (2x-1)\,(3-x+x^2)^{-1/2}\,dx = 2\,(3-x+x^2)^{1/2} + C = 2\sqrt{3-x+x^2} + C$$

Example 4: Determine $\int \frac{36x^2}{(100+x^3)^4}\,dx$ rewrite: $\int 36(100+x^3)^{-4}\,x^2\,dx$

Let $u = 100 + x^3$ (expression inside power $n = -4$)

$du = (3x^2)\,dx$ (derivative of the inside expression)

the adjustment is necessary to replace into the original integral; we get: $x^2\,dx = \frac{1}{3}\,du$

replace $(100+x^3)^{-4}$ with u^{-4} and $x^2\,dx$ with $\frac{1}{3}\,du$; we get:

$$\int 36(100+x^3)^{-4}\,x^2\,dx \longrightarrow \int 36\,u^{-4}\,\tfrac{1}{3}\,du = \int 12\,u^{-4}\,du = 12\,\frac{u^{-3}}{-3} + C = -4\,u^{-3} + C \text{ (power rule is used)}$$

Therefore, replace $u = 100 + x^3$ to get answer:

$$\int 36x^2\,(100+x^3)^{-4}\,dx = -4\,(100+x^3)^{-3} + C = \frac{-4}{(100+x^3)^3} + C$$

New Logarithmic Formula

Given $\displaystyle\int \frac{k}{ax+b}\,dx$ rewrite: $\displaystyle\int k\,(ax+b)^{-1}\,dx$

use substitution: let: $u = ax+b$
$du = a\,dx$ (derivative of the inside expression)

the adjustment is necessary to replace into the original integral; we get: $dx = \frac{1}{a}\,du$

replace $(ax+b)^{-1}$ with u^{-1} and dx with $\frac{1}{a}\,du$; we get:

$$\int k\,(ax+b)^{-1}\,dx \longrightarrow \int k\,u^{-1}\,\tfrac{1}{a}\,du = \int \frac{k}{a}\,u^{-1}\,du = \frac{k}{a}\,\ln|u| + C \quad \text{(log formula is used)}$$

Therefore, replace $u = ax+b$ to get answer:

$$\boxed{\int \frac{k}{ax+b}\,dx = \frac{k}{a}\,\ln|ax+b| + C}$$

Example 5: Determine $\displaystyle\int \frac{6}{5x+4}\,dx$

Use logarithmic formula: $\displaystyle\int \frac{k}{ax+b}\,dx = \frac{k}{a}\,\ln|ax+b| + C$

therefore $\displaystyle\int \frac{6}{5x+4}\,dx = \frac{6}{5}\,\ln|5x+4| + C$

Example 6: Determine $\displaystyle\int \frac{x-6x^3}{x^2-3x^4}\,dx$ rewrite: $\displaystyle\int (x^2-3x^4)^{-1}\,(x-6x^3)\,dx$

Let $u = x^2-3x^4$ (expression inside power $n=-1$)

$du = (2x-12x^3)\,dx = 2(x-6x^3)\,dx$ (derivative of the inside expression)

the adjustment is necessary to replace into the original integral; we get: $(x-6x^3)\,dx = \frac{1}{2}\,du$

replace $(x^2-3x^4)^{-1}$ with u^{-1} and $(x-6x^3)\,dx$ with $\frac{1}{2}\,du$; we get:

$$\int (x^2-3x^4)^{-1}\,(x-6x^3)\,dx \longrightarrow \int u^{-1}\,\tfrac{1}{2}\,du = \tfrac{1}{2}\,\ln|u| + C \quad \text{(log formula is used)}$$

Therefore, replace $u = x^2-3x^4$ to get answer: $\displaystyle\int \frac{x-6x^3}{x^2-3x^4}\,dx = \frac{1}{2}\,\ln|x^2-3x^4| + C$

Example 7: Determine $\int x\,e^{3x^2}\,dx$ rewrite: $\int e^{3x^2}\,x\,dx$

Let $u = 3x^2$ (expression in the exponent of exponential e)

$du = (6x)\,dx$ (derivative of the exponent of e)

the adjustment is necessary to replace into the original integral; we get: $x\,dx = \frac{1}{6}\,du$

replace $3x^2$ with u and $x\,dx$ with $\frac{1}{6}\,du$; we get:

$\int e^{3x^2}\,x\,dx \longrightarrow \int e^u\,\frac{1}{6}\,du = \frac{1}{6}\,e^u + C$ (exponential formula is used)

Therefore, replace $u = 3x^2$ to get answer: $\int x\,e^{3x^2}\,dx = \frac{1}{6}\,e^{3x^2} + C$

Example 8: Determine $\int \frac{e^{\sqrt{x}+4}}{\sqrt{x}}\,dx$ rewrite: $\int e^{4+\sqrt{x}}\,\frac{1}{\sqrt{x}}\,dx$

Let $u = 4 + \sqrt{x}$ (expression in the exponent of exponential e)

$du = \frac{1}{2}\,x^{-1/2}\,dx = \frac{1}{2\sqrt{x}}\,dx$ (derivative of the exponent of e)

the adjustment is necessary to replace into the original integral; we get: $\frac{1}{\sqrt{x}}\,dx = 2\,du$

replace $\sqrt{x}+4$ with u and $\frac{1}{\sqrt{x}}\,dx$ with $2\,du$; we get:

$\int e^{\sqrt{x}+4}\,\frac{1}{\sqrt{x}}\,dx \longrightarrow \int e^u\,2\,du = 2\,e^u + C$ (exponential formula is used)

Therefore, replace $u = \sqrt{x}+4$ to get answer: $\int \frac{e^{\sqrt{x}+4}}{\sqrt{x}}\,dx = 2\,e^{\sqrt{x}+4} + C$

Example 9: Determine $\int \frac{4x}{e^{x^2}}\,dx$ rewrite: $\int 2\,e^{-x^2}\,.\,2x\,dx$

Let $u = -x^2$ (expression in the exponent of exponential e)

$du = (-2x)\,dx$ (derivative of the exponent of e)

the adjustment is necessary to replace into the original integral; we get: $2x\,dx = -\,du$

replace $-x^2$ with u and $2x\,dx$ with $-\,du$; we get:

$\int 2\,e^{-x^2}\,.\,2x\,dx \longrightarrow \int 2\,e^u\,(-1)\,du = -2\,e^u + C$ (exponential formula is used)

Therefore, replace $u = -x^2$ to get answer: $\int 4x\,e^{-x^2}\,dx = -2\,e^{-x^2} + C$

New Exponential Formula Given $\int e^{kx}\,dx$

Let $u = kx$ (expression in the exponent of exponential e)

$du = (k)\,dx$ (derivative of the exponent of e)

the adjustment is necessary to replace into the original integral; we get: $dx = \frac{1}{k}\,du$

replace kx with u and dx with $\frac{1}{k}\,du$; we get:

$$\int e^{kx}\,dx \longrightarrow \int e^{u}\,\tfrac{1}{k}\,du = \tfrac{1}{k}\,e^{u} + C \text{ (exponential formula is used)}$$

Therefore, replace $u = kx$ to get answer:

$$\boxed{\int e^{kx}\,dx = \frac{1}{k}\,e^{kx} + C}$$

Example 10: Given $\int 4\,e^{5x}\,dx$

use exponential formula: $\int e^{kx}\,dx = \dfrac{1}{k}\,e^{kx} + C$

therefore: $\int 4\,e^{5x}\,dx = \dfrac{4}{5}\,e^{5x} + C$

Long Division (Algebra Work)

Long division procedure:

Suppose we have a quotient of polynomials, $\dfrac{P(x)}{Q(x)}$ we must divide if:

degree of polynomial $P(x) \geq$ degree of polynomial $Q(x)$

we cannot divide if: degree of polynomial $P(x) <$ degree of polynomial $Q(x)$

Let degree of $P(x) >$ degree of $Q(x)$, we must then use long division to get an easier way to integrate. Once the long division is done, we get:

$$\boxed{\frac{P(x)}{Q(x)} = \text{result of division} + \frac{\text{remainder}}{Q(x)}}$$

Note: The objective of long division is to make integration easier.

Example 11: (Linear Denominator)

Determine $\displaystyle\int \frac{2x^3 - 6x + 5}{x - 4}\,dx$

Long Division:

$$
\begin{array}{r|l}
 & \quad\; 2x^2 \;+\; 8x \;+\; 26 \\
\hline
x - 4 & 2x^3 \qquad\quad -\; 6x + \;\; 5 \\
 & -(\,2x^3 - 8x^2\,) \\
\hline
 & \qquad 8x^2 - 6x + \;\; 5 \\
 & \qquad -(\,8x^2 - 32x\,) \\
\hline
 & \qquad\qquad 26x + \;\; 5 \\
 & \qquad\qquad -(\,26x - 104\,) \\
\hline
 & \qquad\qquad\qquad 109
\end{array}
$$

then $\displaystyle\int \frac{2x^3 - 6x + 5}{x - 4}\,dx = \int \left(2x^2 + 8x + 26 + \frac{109}{x - 4}\right) dx$

Use power rule $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ and log formula $\displaystyle\int \frac{k}{ax + b}\,dx = \frac{k}{a}\ln|ax + b| + C$ to get:

$$\int \frac{2x^3 - 6x + 5}{x - 4}\,dx = \int \left(2x^2 + 8x + 26 + \frac{109}{x - 4}\right) dx = \tfrac{2}{3}\,x^3 + 4x^2 + 26x + 109\,\ln|x - 4| + C$$

Example 12: (Linear Denominator)

Determine $\displaystyle\int \frac{x - 2}{x + 3}\,dx$

Long Division:

$$
\begin{array}{r|l}
 & \qquad\quad 1 \\
\hline
x + 3 & \qquad x - 2 \\
 & \;\; -(\,x + 3\,) \\
\hline
 & \qquad\quad -5
\end{array}
$$

then $\displaystyle\int \frac{x - 2}{x + 3}\,dx = \int \left(1 + \frac{5}{x + 3}\right) dx$

Use power rule $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ and log formula $\displaystyle\int \frac{k}{ax + b}\,dx = \frac{k}{a}\ln|ax + b| + C$ to get:

$$\int \frac{x - 2}{x + 3}\,dx = \int \left(1 + \frac{5}{x + 3}\right) dx = x + 5\,\ln|x + 3| + C$$

Example 13: (Linear Denominator)

Determine $\int \frac{x^2+4}{x-1}\,dx$ since degree of $P(x)$ is $2 >$ degree of $Q(x)$ is 1,

we must use long division to get an easier way to integrate.

Long Division:

$$\begin{array}{r|l} & x+1 \\ \hline x-1 \;) & x^2 \quad\quad +4 \\ & -(\,x^2 - x\,) \\ \hline & x+4 \\ & -(\,x-1\,) \\ \hline & 5 \end{array}$$

then $\int \frac{x^2+4}{x-1}\,dx = \int \left(x+1+\frac{5}{x-1}\right)dx$

Use power rule $\int x^n\,dx = \frac{x^{n+1}}{n+1}+C$ and log formula $\int \frac{k}{ax+b}\,dx = \frac{k}{a}\ln|ax+b|+C$ to get:

$$\int \frac{x^2+4}{x-1}\,dx = \int \left(x+1+\frac{5}{x-1}\right)dx = \tfrac{1}{2}\,x^2+x+5\,\ln|x-1|+C$$

Example 14: (Linear Denominator)

Determine $\int \frac{x^3+2x-1}{x+4}\,dx$ since degree of $P(x)$ is $3 >$ degree of $Q(x)$ is 1,

we must use long division to get an easier way to integrate.

Long Division:

$$\begin{array}{r|l} & x^2 - 4x + 18 \\ \hline x+4 \;) & x^3 \quad + 2x - 1 \\ & -(\,x^3+4x^2\,) \\ \hline & -4x^2 + 2x - 1 \\ & -(\,-4x^2 - 16x\,) \\ \hline & 18x - 1 \\ & -(\,18x+72\,) \\ \hline & -73 \end{array}$$

then $\int \frac{x^3+2x-1}{x+4}\,dx = \int \left(x^2-4x+18+\frac{-73}{x+4}\right)dx$

Use power rule $\int x^n\,dx = \frac{x^{n+1}}{n+1}+C$ and log formula $\int \frac{k}{ax+b}\,dx = \frac{k}{a}\ln|ax+b|+C$ to get:

$$\int \frac{x^3+2x-1}{x+4}\,dx = \int \left(x^2-4x+18+\frac{-73}{x+4}\right)dx = \tfrac{1}{3}\,x^3-2x^2+18x-73\,\ln|x+4|+C$$

Example 15: (Quadratic Denominator)

Determine $\displaystyle\int \frac{2x^3+6x^2+12x-3}{x^2+3x+7}\,dx$ since degree of $P(x)$ is $3>$ degree of $Q(x)$ is 2,

we must use long division to get an easier way to integrate.

Long Division:

$$\begin{array}{r|l} & \quad 2x \\ x^2+3x+7 & \; 2x^3+6x^2+12x-3 \\ & -(2x^3+6x^2+14x\;) \\ \hline & \qquad\qquad -\;2x-3 \end{array}$$

then $\displaystyle\int \frac{2x^3+6x^2+12x-3}{x^2+3x+7}\,dx = \int \left(2x + \frac{-2x-3}{x^2+3x+7}\right) dx$

Use substitution for remaining fraction: Let $u = x^2+3x+7 \to du = (2x+3)\,dx \to -du = (-2x-3)\,dx$

$\displaystyle\int \frac{-2x-3}{x^2+3x+7}\,dx \longrightarrow \int \frac{-du}{u} = -\ln|u| + C$ and

therefore $\displaystyle\int \frac{2x^3+6x^2+12x-3}{x^2+3x+7}\,dx = x^2 - \ln|x^2+3x+7| + C$

Example 16: (Quadratic Denominator)

Determine $\displaystyle\int \frac{2x^3+10x}{x^2+7}\,dx$ since degree of $P(x)$ is $3>$ degree of $Q(x)$ is 2,

we must use long division to get an easier way to integrate.

Long Division:

$$\begin{array}{r|l} & \quad 2x \\ x^2+7 & \; 2x^3\;+10x \\ & -(2x^3\;+14x\,) \\ \hline & \qquad -\;4x \end{array}$$

then $\displaystyle\int \frac{2x^3+10x}{x^2+7}\,dx = \int \left(2x + \frac{-4x}{x^2+7}\right) dx$

Use substitution for remaining fraction: Let $u = x^2+7 \to du = (2x)\,dx \to -2\,du = (-4x)\,dx$

$\displaystyle\int \frac{-4x}{x^2+7}\,dx \longrightarrow \int \frac{-2\,du}{u} = -2\,\ln|u| + C$ and

therefore $\displaystyle\int \frac{2x^3+10x}{x^2+7}\,dx = x^2 - 2\,\ln(x^2+7) + C$

Example 17(a): Determine $\int \frac{4}{2+\sqrt{x}}\,dx$ First type of Algebraic substitution

Let $u = \sqrt{x} = x^{1/2}$ (radical term) ; $du = \frac{1}{2}\,x^{-1/2}\,dx = \frac{1}{2\sqrt{x}}\,dx$ (derivative of u)

replace in du $\sqrt{x}$ with $u \to du = \frac{1}{2u}\,dx \to 2u\,du = dx$

the adjustment is necessary to replace into the original integral;

replace $2+\sqrt{x}$ with $2+u$ and dx with $2u\,du$; we get:

$$\int \frac{4}{2+\sqrt{x}}\,dx \longrightarrow \int \frac{8u}{2+u}\,du$$ must use long division:

$$\begin{array}{r|l} & 8 \\ u+2 & 8u \\ & -(8u+16) \\ & -16 \end{array}$$

$$\int \frac{8u}{2+u}\,du = \int \left(8 + \frac{-16}{2+u}\right) du = 8u - 16\,\ln|2+u| + C$$

Therefore, replace $u = \sqrt{x}$ to get answer: $\int \frac{4}{2+\sqrt{x}}\,dx = 8\sqrt{x} - 16\,\ln\left(2+\sqrt{x}\right) + C$

Example 17(b): Determine $\int \frac{4}{2+\sqrt{x}}\,dx$ Second type of Algebraic substitution

Let $u = 2+\sqrt{x} = 2 + x^{1/2}$ (expression with radical term) ; $du = \frac{1}{2}\,x^{-1/2}\,dx = \frac{1}{2\sqrt{x}}\,dx$ (derivative of u)

replace in du $\sqrt{x}$ with $u-2 \to du = \frac{1}{2(u-2)}\,dx \to 2(u-2)\,du = dx$

the adjustment is necessary to replace into the original integral;

replace $2+\sqrt{x}$ with u and dx with $2(u-2)\,du$; we get:

$$\int \frac{4}{2+\sqrt{x}}\,dx \longrightarrow \int \frac{(4)\,2(u-2)}{u}\,du = \int (8u-16)u^{-1}\,du = \int \left(8 - 16u^{-1}\right)\,du = 8u - 16\,\ln|u| + C$$

Therefore, replace $u = 2+\sqrt{x}$ to get answer: $\int \frac{4}{2+\sqrt{x}}\,dx = 8\left(2+\sqrt{x}\right) - 16\,\ln\left(2+\sqrt{x}\right) + C$

$$\int \frac{4}{2+\sqrt{x}}\,dx = 8\sqrt{x} - 16\,\ln\left(2+\sqrt{x}\right) + K \quad \text{where } K = 16 + C$$

Example 18(a): Determine $\displaystyle\int \frac{5\sqrt{x}}{\sqrt{x}+3}\,dx$ First type of Algebraic substitution

Let $u = \sqrt{x} = x^{1/2}$ (radical term) ; $du = \frac{1}{2}\,x^{-1/2}\,dx = \dfrac{1}{2\sqrt{x}}\,dx$ (derivative of u)

replace in du $\sqrt{x}$ with $u \to du = \dfrac{1}{2u}\,dx \to 2u\,du = dx$

the adjustment is necessary to replace into the original integral;

replace $\sqrt{x}+3$ with $u+3$ and dx with $2u\,du$; we get:

$$\int \frac{5\sqrt{x}}{\sqrt{x}+3}\,dx \longrightarrow \int \frac{5u\,.\,2u}{u+3}\,du = \int \frac{10u^2}{u+3}\,du \qquad \text{must use long division:}$$

$$\begin{array}{r l}
 & \quad 10u - 30 \\
u+3\ \big) & 10u^2 \\
 & -(\,10u^2 + 30u\,) \\
 & \quad -30u \\
 & \quad -(\,-30u - 90\,) \\
 & \qquad\quad 90
\end{array}$$

$$\int \frac{10u^2}{u+3}\,du = \int \left(10u - 30 + \frac{90}{u+3}\right) du = 5u^2 - 30u + 90\,\ln|u+3| + C$$

Therefore, replace $u = \sqrt{x}$ to get answer: $\displaystyle\int \frac{5\sqrt{x}}{\sqrt{x}+3}\,dx = 5x - 30\sqrt{x} + 90\,\ln\left(\sqrt{x}+3\right) + C$

Example 18(b): Determine $\displaystyle\int \frac{5\sqrt{x}}{\sqrt{x}+3}\,dx$ Second type of Algebraic substitution

Let $u = \sqrt{x}+3 = x^{1/2}+3$ (expression with radical term) ; $du = \frac{1}{2}\,x^{-1/2}\,dx = \dfrac{1}{2\sqrt{x}}\,dx$ (derivative of u)

replace in du $\sqrt{x}$ with $u-3 \to du = \dfrac{1}{2(u-3)}\,dx \to 2(u-3)\,du = dx$

the adjustment is necessary to replace into the original integral;

replace $\sqrt{x}+3$ with u and dx with $(2u-6)\,du$; we get:

$$\int \frac{5\sqrt{x}}{\sqrt{x}+3}\,dx \longrightarrow \int \frac{5(u-3)\,(2u-6)}{u}\,du = \int (10u^2 - 60u + 90)u^{-1}\,du$$

$$\int \left(10u - 60 + 90u^{-1}\right)\,du = 5u^2 - 60u + 90\,\ln|u| + C$$

Therefore, replace $u = \sqrt{x}+3$ to get answer: $\displaystyle\int \frac{5\sqrt{x}}{\sqrt{x}+3}\,dx = 5\left(\sqrt{x}+3\right)^2 - 60\left(\sqrt{x}+3\right) + 90\,\ln\left(\sqrt{x}+3\right) + C$

$$5\,(x + 6\sqrt{x} + 9\,) - 60\sqrt{x} - 180 + 90\,\ln(\sqrt{x}+3\,) + C = 5x + 30\sqrt{x} + 45 - 60\sqrt{x} - 180 + 90\,\ln(\sqrt{x}+3\,) + C$$

$$= 5x - 30\sqrt{x} + 90\,\ln(\sqrt{x}+3\,) + K \quad \text{where } K = 45 - 180 + C$$

<u>Example 19:</u> Determine $\int \frac{3\,e^x}{e^x+4}\,dx$; rewrite $\int \frac{3}{e^x+4}\,e^x\,dx$

Let $u = e^x$ (exponential term) ; $du = e^x\,dx$ (derivative of u)

no adjustment is necessary to replace into the original integral;

replace e^x with u and $e^x\,dx$ with du ; we get:

$$\int \frac{3\,e^x}{e^x+4}\,dx \longrightarrow \int \frac{3}{u+4}\,du = 3\,\ln|u+4| + C \quad \text{(log formula is used)}$$

Therefore, replace $u = e^x$ to get answer: $\int \frac{3\,e^x}{e^x+4}\,dx = 3\,\ln(e^x+4) + C$

<u>Example 20:</u> Determine $\int \frac{e^{5x}+5}{e^{2x}}\,dx$; rewrite $\int \left(e^{5x}+5\right)e^{-2x}\,dx$

multiply $\int \left(e^{3x}+5\,e^{-2x}\right)\,dx$

use exponential formula: $\int e^{kx}\,dx = \frac{1}{k}\,e^{kx} + C$

Therefore, $\int \left(e^{3x}+5\,e^{-2x}\right)\,dx = \frac{1}{3}\,e^{3x} + \frac{5}{-2}\,e^{-2x} + C = \frac{1}{3}\,e^{3x} - \frac{5}{2}\,e^{-2x} + C$

<u>Example 21:</u> Determine $\int \frac{e^x\left(e^{2x}+6\right)}{e^x+2}\,dx$

Let $u = e^x$ (exponential term) ; $du = e^x\,dx$ (derivative of u) ; $u^2 = e^{2x}$

no adjustment is necessary to replace into the original integral;

replace e^x with u ; e^{2x} with u^2 and $e^x\,dx$ with du ; we get:

$$\int \frac{e^x\left(e^{2x}+6\right)}{e^x+2}\,dx \longrightarrow \int \frac{u^2+6}{u+2}\,du \qquad \text{must use long division:}$$

$$\begin{array}{r|l} & u - 2 \\ u+2 & u^2 \qquad + 6 \\ & -(\,u^2 + 2u\,) \\ \hline & -2u + 6 \\ & -(\,-2u - 4\,) \\ \hline & 10 \end{array}$$

$$\int \frac{u^2+6}{u+2}\,du = \int \left(u - 2 + \frac{10}{u+2}\right)du = \frac{1}{2}\,u^2 - 2u + 10\,\ln|u+2| + C$$

Therefore, replace $u = e^x$ to get answer: $\int \frac{e^x\left(e^{2x}+6\right)}{e^x+2}\,dx = \frac{1}{2}\,e^{2x} - 2\,e^x + 10\,\ln(e^x+2) + C$