What is the Antiderivative?

In a derivative problem, a function $f(x)$ is given and you find the derivative $f'(x)$ using the formulas and rules of derivatives shown in a previous tutorial.

In an antiderivative problem, the derivative $f'(x)$ is given and you find a function $f(x)$ using the formulas of antiderivatives shown later in this tutorial.

In other words, the antiderivative is the reverse question to the derivative question.

Trigonometric Antiderivatives Formulas

$$\boxed{\int \sin(x)\,dx = -\cos(x) + C} \qquad \boxed{\int \cos(x)\,dx = \sin(x) + C}$$

$$\boxed{\int \sec^2(x)\,dx = \tan(x) + C} \qquad \boxed{\int \csc^2(x)\,dx = -\cot(x) + C}$$

These formulas are verified if the derivative is applied to each antiderivative answer.

Sum, Difference & Division of terms

To find the antiderivative of a sum or a difference of terms, use appropriate trigonometric formulas for each term.

Example 1: Determine $\displaystyle\int \Big(3\,\cos(x) - 4\,\sin(x) + 5\,\sec^2(x)\Big)\,dx$

$$= 3\Big(\sin(x)\Big) - 4\Big(-\cos(x)\Big) + 5\Big(\tan(x)\Big) + C == 3\,\sin(x) + 4\,\cos(x) + 5\,\tan(x) + C$$

Example 2: Determine $\displaystyle\int \frac{2\,\sec\,x - 3\,\cos\,x}{\cos\,x}\,dx \longrightarrow$ Rewrite: $\displaystyle\int 2\,\frac{\sec\,x}{\cos\,x}\,dx - \int 3\,\frac{\cos\,x}{\cos\,x}\,dx$

$$= \int 2\,\sec\,x\,.\,\sec\,x\,dx - \int 3\,(1)\,dx = \int 2\,\sec^2\,x\,dx - \int 3\,dx = 2\,\tan\,x - 3x + C$$

Example 3: Determine $\displaystyle\int \frac{4x\,\sin\,x - 5\,\csc\,x}{\sin\,x}\,dx \longrightarrow$ Rewrite: $\displaystyle\int 4x\,\frac{\sin\,x}{\sin\,x}\,dx - \int 5\,\frac{\csc\,x}{\sin\,x}\,dx$

$$= \int 4x\,(1)\,dx - \int 5\,\csc\,x\,.\,\csc\,x\,dx = \int 4x\,dx - \int 5\,\csc^2\,x\,dx$$

$$= 2x^2 - 5\,(-\cot\,x) + C = 2x^2 + 5\,\cot\,x + C$$

Example 4: Determine $\int \dfrac{5 - 2\sin^3 x}{\sin^2 x}\,dx \longrightarrow$ Rewrite: $\int 5\dfrac{1}{\sin^2 x}\,dx - \int 2\,\dfrac{\sin^3 x}{\sin^2 x}\,dx$

$= \int 5\csc^2 x\,dx - \int 2\sin x\,dx = 5\,(-\cot x) - 2\,(-\cos x) + C = -5\cot x + 2\cos x + C$

Example 5: Determine $\int \dfrac{3\cot x + 4x\sin x\cos x}{\sin x\cos x}\,dx \longrightarrow$ Rewrite: $\int 3\dfrac{\frac{\cos x}{\sin x}}{\sin x\cos x}\,dx + \int 4x\,\dfrac{\sin x\cos x}{\sin x\cos x}\,dx$

$= \int 3\dfrac{\cos x}{\sin x}\cdot\dfrac{1}{\sin x\cos x}\,dx + \int 4x\,(1)\,dx = \int 3\dfrac{1}{(\sin x)^2}\,dx + \int 4x\,dx = \int 3\csc^2 x\,dx + \int 4x\,dx$

$= 3\,(-\cot x) + 4\,\dfrac{x^2}{2} + C = -3\cot x + 2x^2 + C$

Chain Rule Substitution

To find the antiderivative of a composite trigonometric function, use chain rule substitution and appropriate trigonometric formulas.

Example 6: Determine $\int 3\cos 4x\,dx = \int 3\cos(4x)\,dx$

Let $u = 4x$ (expression inside of trigonometric function is the angle)

$du = 4\,dx$ (derivative of the inside expression)

the adjustment is necessary to replace into the original integral; we get: $dx = \frac{1}{4}\,du$

replace $\cos(4x)$ with $\cos(u)$ and dx with $\frac{1}{4}\,du$; we get:

$\int 3\cos(4x)\,dx \longrightarrow \int 3\cos(u)\,\frac{1}{4}\,du = \int \frac{3}{4}\cos(u)\,du = \frac{3}{4}\sin(u) + C$ (trigonometric formula is used)

Therefore, replace $u = 4x$ to get answer:

$\int 3\cos(4x)\,dx = \frac{3}{4}\sin(4x) + C$

<u>Example 7</u>: Determine $\int 6\,\sin{(2x+1)}\,dx$

Let $u = 2x+1$ (expression inside of trigonometric function is the angle)

$du = 2\,dx$ (derivative of the inside expression)

the adjustment is necessary to replace into the original integral; we get: $dx = \frac{1}{2}\,du$

replace $\sin(2x+1)$ with $\sin(u)$ and dx with $\frac{1}{2}\,du$; we get:

$\int 6\,\sin(2x+1)\,dx \longrightarrow \int 6\,\sin(u)\,\frac{1}{2}\,du = \int 3\,\sin(u)\,du = -3\,\cos(u) + C$ (trigonometric formula is used)

Therefore, replace $u = 2x+1$ to get answer: $\int 6\,\sin(2x+1)\,dx = -3\,\cos(2x+1) + C$

<u>Example 8</u>: Determine $\int 4\,\sec^2{(8x+3)}\,dx$

Let $u = 8x+3$ (expression inside of trigonometric function is the angle)

$du = 8\,dx$ (derivative of the inside expression)

the adjustment is necessary to replace into the original integral; we get: $dx = \frac{1}{8}\,du$

replace $\sec^2(8x+3)$ with $\sec^2(u)$ and dx with $\frac{1}{8}\,du$; we get:

$\int 4\,\sec^2(8x+3)\,dx \longrightarrow \int 4\,\sec^2(u)\,\frac{1}{8}\,du = \int \frac{1}{2}\,\sec^2(u)\,du = \frac{1}{2}\,\tan(u) + C$ (trigonometric formula is used)

Therefore, replace $u = 8x+3$ to get answer: $\int 4\,\sec^2(8x+3)\,dx = \frac{1}{2}\,\tan(8x+3) + C$

<u>Example 9</u>: Determine $\int 4x\,\sin{(x^2)}\,dx$

Let $u = x^2$ (expression inside of trigonometric function is the angle)

$du = 2x\,dx$ (derivative of the inside expression)

the adjustment is necessary to replace into the original integral; we get: $4x\,dx = 2\,du$

replace $\sin(x^2)$ with $\sin(u)$ and $4x\,dx$ with $2\,du$; we get:

$\int 4x\,\sin(x^2)\,dx \longrightarrow \int \sin(u)\,2\,du = \int 2\,\sin(u)\,du = -2\,\cos(u) + C$ (trigonometric formula is used)

Therefore, replace $u = x^2$ to get answer: $\int 4x\,\sin(x^2)\,dx = -2\,\cos(x^2) + C$

<u>Example 10:</u> Determine $\int 3\,e^x\ \cos\left(e^x+2\right)\ dx$

Let $u=e^x+2$ (expression inside of trigonometric function is the angle)

$du=e^x\,dx$ (derivative of the inside expression)

the adjustment is necessary to replace into the original integral; we get: $3\,e^x\,dx=3\,du$

replace $\cos\left(e^x+2\right)$ with $\cos(u)$ and $3\,e^x\,dx$ with $3\,du$; we get:

$$\int 3\,e^x\ \cos\left(e^x+2\right)\ dx \longrightarrow \int \cos(u)\,3\,du = \int 3\,\cos(u)\,du = 3\,\sin(u)+C \ \text{ (trigonometric formula is used)}$$

Therefore, replace $u=e^x+2$ to get answer: $\int 3\,e^x\ \cos\left(e^x+2\right)\ dx = 3\,\sin\left(e^x+2\right)+C$

<u>Tangent Formula</u> Determine $\int \tan(x)\,dx = \int \dfrac{\sin(x)}{\cos(x)}\,dx = \int \sin\,x\,.\,(\cos\,x)^{-1}\ dx$

Let $u=\cos(x)$ (expression inside of power function)

$du=-\sin\,x\,dx$ (derivative of the inside expression)

the adjustment is necessary to replace into the original integral; we get: $\sin\,x\,dx=-\,du$

replace $\cos\,x$ with u and $\sin\,x\,dx$ with $-\,du$; we get:

$$\int \tan(x)\,dx \longrightarrow \int \left(u^{-1}\right)(-1)\,du = \int -\,u^{-1}\,du = -\ln|u|+C \ \text{ (log formula is used)}$$

Therefore, replace $u=\cos\,x$ to get answer: $\int \tan\,x\,dx = -\ln|\cos\,x|+C = \ln\left|(\cos\,x)^{-1}\right|+C$

remember the ratio: $(\cos\,x)^{-1} = \dfrac{1}{\cos\,x} = \sec\,x$

$$\boxed{\int \tan\,x\,dx = -\ln|\cos\,x|+C = \ln|\sec\,x|+C}$$

Cotangent Formula Determine $\int \cot(x)\,dx = \int \frac{\cos(x)}{\sin(x)}\,dx = \int \cos x\,.\,(\sin x)^{-1}\,dx$

Let $u = \sin(x)$ (expression inside of power function)

$du = \cos x\,dx$ (derivative of the inside expression)

no adjustment is necessary to replace into the original integral;

replace $\sin x$ with u and $\cos x\,dx$ with du ; we get:

$\int \cot(x)\,dx \longrightarrow \int (u^{-1})\,du = \int u^{-1}\,du = \ln|u| + C$ (log formula is used)

Therefore, replace $u = \sin x$ to get answer: $\int \cot x\,dx = \ln|\sin x| + C$

$$\boxed{\int \cot x\,dx = \ln|\sin x| + C}$$

Secant Formula Determine $\int \sec(x)\,dx \longrightarrow$ rewrite $\int \sec x\,\frac{\sec x + \tan x}{\sec x + \tan x}\,dx = \int \frac{\sec^2 x + \sec x\,\tan x}{\sec x + \tan x}\,dx$

$\int \left(\sec^2 x + \sec x\,\tan x\right)\,(\sec x + \tan x)^{-1}\,dx$

Let $u = \sec x + \tan x$ (expression inside of power function)

$du = (\sec x\,\tan x + \sec^2 x)\,dx$ (derivative of the inside expression)

no adjustment is necessary to replace into the original integral;

replace $\sec x + \tan x$ with u and $(\sec x\,\tan x + \sec^2 x)\,dx$ with du ; we get:

$\int \sec(x)\,dx \longrightarrow \int (u^{-1})\,du = \int u^{-1}\,du = \ln|u| + C$ (log formula is used)

Therefore, replace $u = \sec x + \tan x$ to get answer: $\int \sec x\,dx = \ln|\sec x + \tan x| + C$

$$\boxed{\int \sec x\,dx = \ln|\sec x + \tan x| + C}$$

<u>Cosecant Formula</u> Determine $\int \csc(x)\,dx \longrightarrow$ rewrite $\int \csc\, x\, \frac{\csc\, x + \cot\, x}{\csc\, x + \cot\, x}\,dx = \int \frac{\csc^2\, x + \csc\, x\, \cot\, x}{\csc\, x + \cot\, x}\,dx$

$\int \left(\csc^2\, x + \csc\, x\, \cot\, x\right)\, \left(\csc\, x + \cot\, x\right)^{-1}\, dx$

Let $u = \csc\, x + \cot\, x$ (expression inside of power function)

$du = (-\csc\, x\, \cot\, x - \csc^2\, x)\,dx$ (derivative of the inside expression)

the adjustment is necessary to replace into the original integral; we get: $(\csc\, x\, \cot\, x + \csc^2\, x)\,dx = -\,du$

replace $\csc\, x + \cot\, x$ with u and $(\csc\, x\, \cot\, x + \csc^2\, x)\,dx$ with $-du$; we get:

$\int \csc(x)\,dx \longrightarrow \int (u^{-1})\,(-1)\,du = \int -u^{-1}\,du = -\ln|u| + C$ (log formula is used)

Therefore, replace $u = \csc\, x + \cot\, x$ to get answer: $\int \csc\, x\,dx = -\ln|\csc\, x + \cot\, x| + C$

$$\boxed{\int \csc\, x\,dx = -\ln|\csc\, x + \cot\, x| + C = \ln|\csc\, x - \cot\, x| + C}$$

<u>Summary of Trigonometric Integrals Formulas</u>

$\boxed{\int \sin(x)\,dx = -\cos(x) + C}$

$\boxed{\int \cos(x)\,dx = \sin(x) + C}$

$\boxed{\int \sec^2(x)\,dx = \tan(x) + C}$

$\boxed{\int \csc^2(x)\,dx = -\cot(x) + C}$

$\boxed{\int \tan(x)\,dx = \ln|\sec(x)| + C}$

$\boxed{\int \cot(x)\,dx = \ln|\sin(x)| + C}$

$\boxed{\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C}$

$\boxed{\int \csc(x)\,dx = \ln|\csc(x) - \cot(x)| + C}$

short-cut integration formulas

$\boxed{\int \sin(kx)\,dx = -\frac{1}{k}\cos(kx) + C}$

$\boxed{\int \cos(kx)\,dx = \frac{1}{k}\sin(kx) + C}$

$\boxed{\int \sec^2(kx)\,dx = \frac{1}{k}\tan(kx) + C}$

$\boxed{\int \csc^2(kx)\,dx = -\frac{1}{k}\cot(kx) + C}$

$\boxed{\int \tan(kx)\,dx = \frac{1}{k}\ln|\sec(kx)| + C}$

$\boxed{\int \cot(kx)\,dx = \frac{1}{k}\ln|\sin(kx)| + C}$

$\boxed{\int \sec(kx)\,dx = \frac{1}{k}\ln|\sec(kx) + \tan(kx)| + C}$

$\boxed{\int \csc(kx)\,dx = \frac{1}{k}\ln|\csc(kx) - \cot(kx)| + C}$

These formulas are verified if the derivative is applied to each antiderivative answer.

Product of terms

To find the antiderivative of a product of terms, must use the integration by parts technique, use appropriate trigonometric formulas and reduce the answer if necessary.

Example 11: Determine $\displaystyle\int 3x\,\sin(2x)\,dx$

use power rule for derivative and for integration use the substitution technique:

integration by parts formula: $\displaystyle\int u\,.\,dv = u\,.\,v - \int v\,.\,du$

choose $u = 3x \to du = 3\,dx$ and $dv = \sin(2x)\,dx \to v = -\frac{1}{2}\,\cos(2x)$

To get v from dv : choose $u = 2x$, $du = 2\,dx$, $\frac{1}{2}\,du = dx \to \displaystyle\int \frac{1}{2}\,\sin(u)\,du = -\frac{1}{2}\,\cos(u) + C$

Integration by Parts gives: $\displaystyle\int 3x\,\sin(2x)\,dx = 3x\,.\,\left(-\frac{1}{2}\right)\,\cos(2x) - \int -\frac{1}{2}\,\cos(2x)\,.\,3\,dx$

Notice that the integral on right side of the equation is easier to solve than the integral on the left side of the equation.

dv always has dx in the setup choice.

To solve $\displaystyle\int -\frac{3}{2}\,\cos(2x)\,dx$: choose same substitution:

$u = 2x$, $du = 2\,dx$, $\frac{1}{2}\,du = dx \to \displaystyle\int -\frac{3}{2}\,\cos(u)\,\frac{1}{2}\,du = -\frac{3}{4}\,\sin(u) + C$

$$\int 3x\,\sin(2x)\,dx = 3x\,.\,\left(-\frac{1}{2}\right)\,\cos(2x) - \left[-\frac{3}{4}\,\sin(2x)\right] + C = -\frac{3}{2}\,x\,\cos(2x) + \frac{3}{4}\,\sin(2x) + C$$

<u>Example 12:</u> Determine $\int (4x-1)\,\cos(3x)\,dx$

use power rule for derivative and for integration use the substitution technique:

integration by parts formula: $\int u\,.\,dv = u\,.\,v - \int v\,.\,du$

choose $u = 4x-1 \rightarrow du = 4\,dx$ and $dv = \cos(3x)\,dx \rightarrow v = \frac{1}{3}\,\sin(3x)$

To get v from dv : choose $u = 3x$, $du = 3\,dx$, $\frac{1}{3}\,du = dx \rightarrow \int \frac{1}{3}\,\cos(u)\,du = \frac{1}{3}\,\sin(u) + C$

Integration by Parts gives: $\int (4x-1)\,\cos(3x)\,dx = (4x-1)\,.\,\left(\frac{1}{3}\right)\,\sin(3x) - \int \frac{1}{3}\,\sin(3x)\,.\,4\,dx$

Notice that the integral on right side of the equation is easier to solve than the integral on the left side of the equation.

dv always has dx in the setup choice.

To solve $\int \frac{4}{3}\,\sin(3x)\,dx$: choose same substitution:

$u = 3x$, $du = 3\,dx$, $\frac{1}{3}\,du = dx \rightarrow \int \frac{4}{3}\,\sin(u)\,\frac{1}{3}\,du = -\frac{4}{9}\,\cos(u) + C$

$\int (4x-1)\,\cos(3x)\,dx = (4x-1)\,.\,\left(\frac{1}{3}\right)\,\sin(3x) - \left[-\frac{4}{9}\,\cos(3x)\right] + C = \frac{1}{3}\,(4x-1)\,\sin(3x) + \frac{4}{9}\,\cos(3x) + C$

<u>Example 13:</u> Determine $M = \int e^{-x} \sin(2x)\, dx$

integration by parts formula: $\int u \,.\, dv = u \,.\, v - \int v \,.\, du$

choose $u = e^{-x} \rightarrow du = -e^{-x}\, dx$ and $dv = \sin(2x)\, dx \rightarrow v = -\frac{1}{2} \cos(2x)$

To find v from dv : use substitution $u = 2x$ and trigonometric formula $\int \sin(u)\, du$

Integration by Parts gives: $M = \int e^{-x} \sin(2x)\, dx = -\frac{1}{2} e^{-x} \cos(2x) - \int \frac{1}{2} e^{-x} \cos(2x)\, dx$

Repeat integration by parts: choose $u = e^{-x} \rightarrow du = -e^{-x}\, dx$ and $dv = \cos(2x)\, dx \rightarrow v = \frac{1}{2} \sin(2x)$

To find v from dv : use substitution $u = 2x$ and trigonometric formula $\int \cos(u)\, du$, we get:

$$M = \int e^{-x} \sin(2x)\, dx = -\frac{1}{2} e^{-x} \cos(2x) - \left[\frac{1}{4} e^{-x} \sin(2x) - \int -\frac{1}{4} e^{-x} \sin(2x)\, dx\right]$$

$$M = \int e^{-x} \sin(2x)\, dx = -\frac{1}{2} e^{-x} \cos(2x) - \frac{1}{4} e^{-x} \sin(2x) - \frac{1}{4} M$$

$$\frac{5}{4} M = -\frac{1}{2} e^{-x} \cos(2x) - \frac{1}{4} e^{-x} \sin(2x)$$

$$M = \int e^{-x} \sin(2x)\, dx = -\frac{2}{5} e^{-x} \cos(2x) - \frac{1}{5} e^{-x} \sin(2x) + C$$

Example 14: Determine $\int x^3 \sin(4x)\,dx$

must use integration by parts formula three times.

Tabular form (use for repeated integration by parts):

The table ends when the derivatives left column has 0 . The right column represents antiderivatives.
Also every row has a sign that alternates.

	u	dv
$+$	x^3	$\sin(4x)$
$-$	$3x^2$	$-\frac{1}{4}\cos(4x)$
$+$	$6x$	$-\frac{1}{16}\sin(4x)$
$-$	6	$\frac{1}{64}\cos(4x)$
	0	$\frac{1}{256}\sin(4x)$

$$\int x^3 \sin(4x)\,dx = +x^3\left(-\frac{1}{4}\cos(4x)\right) - 3x^2\left(-\frac{1}{16}\sin(4x)\right) + 6x\left(\frac{1}{64}\cos(4x)\right) - 6\left(\frac{1}{256}\sin(4x)\right) + C$$

$$\int x^3 \sin(4x)\,dx = -\frac{1}{4}x^3\cos(4x) + \frac{3}{16}x^2\sin(4x) + \frac{3}{32}x\cos(4x) - \frac{3}{128}\sin(4x) + C$$

Example 15: Determine $\int (5x^3+2)\cos(2x)\,dx$

must use integration by parts formula three times.

Tabular form (use for repeated integration by parts):

The table ends when the derivatives left column has 0 . The right column represents antiderivatives.
Also every row has a sign that alternates.

	u	dv
$+$	$(5x^3+2)$	$\cos(2x)$
$-$	$15x^2$	$\frac{1}{2}\sin(2x)$
$+$	$30x$	$-\frac{1}{4}\cos(2x)$
$-$	30	$-\frac{1}{8}\sin(2x)$
	0	$\frac{1}{16}\cos(2x)$

$$\int (5x^3+2)\cos(2x)\,dx = +(5x^3+2)\left(\frac{1}{2}\sin(2x)\right) - 15x^2\left(-\frac{1}{4}\cos(2x)\right) + 30x\left(-\frac{1}{8}\sin(2x)\right) - 30\left(\frac{1}{16}\cos(2x)\right) + C$$

$$\int (5x^3+2)\cos(2x)\,dx = \frac{5}{2}x^3\sin(2x) + \frac{15}{4}x^2\cos(2x) - \frac{15}{4}x\sin(2x) - \frac{15}{8}\cos(2x) + \sin(2x) + C$$